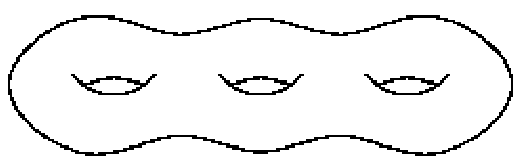
Free Actions on Handlebodies



handlebody = (compact) 3-dimensional orientable handlebody

action = effective action of a finite group G on a handlebody, by orientation-preserving (smooth- or PL-) homeomorphisms

Actions on handlebodies have been extensively studied. See articles by various combinations of: Bruno Zimmermann, Andy Miller, John Kalliongis, McC.

Those articles examine the general case of actions that are not necessarily free. The first focus on *free* actions seems to be:

J. H. Przytycki, Free actions of $\mathbb{Z}_n$ on handlebodies, *Bull. Acad. Polonaise des Sciences* XXVI (1978), 617-624.

The remainder of this talk concerns recent joint work with **Marcus Wanderley,** of Universidade Federal de Pernambuco, Brazil.

Elementary Observation: Every finite group acts freely on a handlebody.

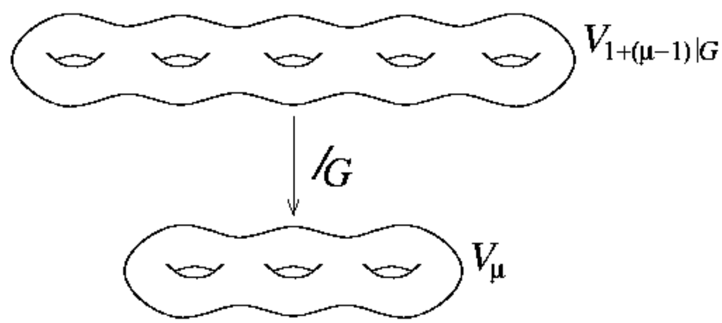
Proof: Let V_μ be a handlebody of genus μ, where μ is the minimum number of elements in a generating set for G.

Since $\pi_1(V_\mu)$ is free of rank μ, there is a surjective homomorphism $\phi\colon \pi_1(V_\mu) \to G$.

The covering of V_μ corresponding to the kernel of ϕ is a handlebody (since its fundamental group is free), and it admits an action by G by covering transformations, with quotient V_μ. $\square$

$$\chi \quad \Rightarrow \quad \text{this covering is } V_{1+(\mu-1)|G|}.$$

There is a simple *stabilization* process for going from an action of G on $V_{1+(\mu-1)|G|}$ to an action on $V_{1+(\mu-1)|G|+|G|}$.



Adding a small 1-handle to the quotient handlebody corresponds to adding $|G|$ small 1-handles to $V_{1+(\mu-1)|G|}$, which are permuted by the action of G. The result is a free G-action on $V_{1+(\mu-1)|G|+|G|}$.

Repeating, we see that G acts freely on the handlebodies $V_{1+(\mu+k-1)|G|}$ for all $k \geq 0$, and Euler characteristic considerations show that these are the only genera that admit free G-actions.

Two actions $\phi, \psi \colon G \to \text{Homeo}(V)$ are *equivalent* when they are the same after a change of coordinates on V.

(That is, there exists a homeomorphism h of V so that $\phi(g) = h \circ \psi(g) \circ h^{-1}$ for all $g \in G$.)

They are *weakly equivalent* when they are equivalent after changing one of them by an automorphism of G.

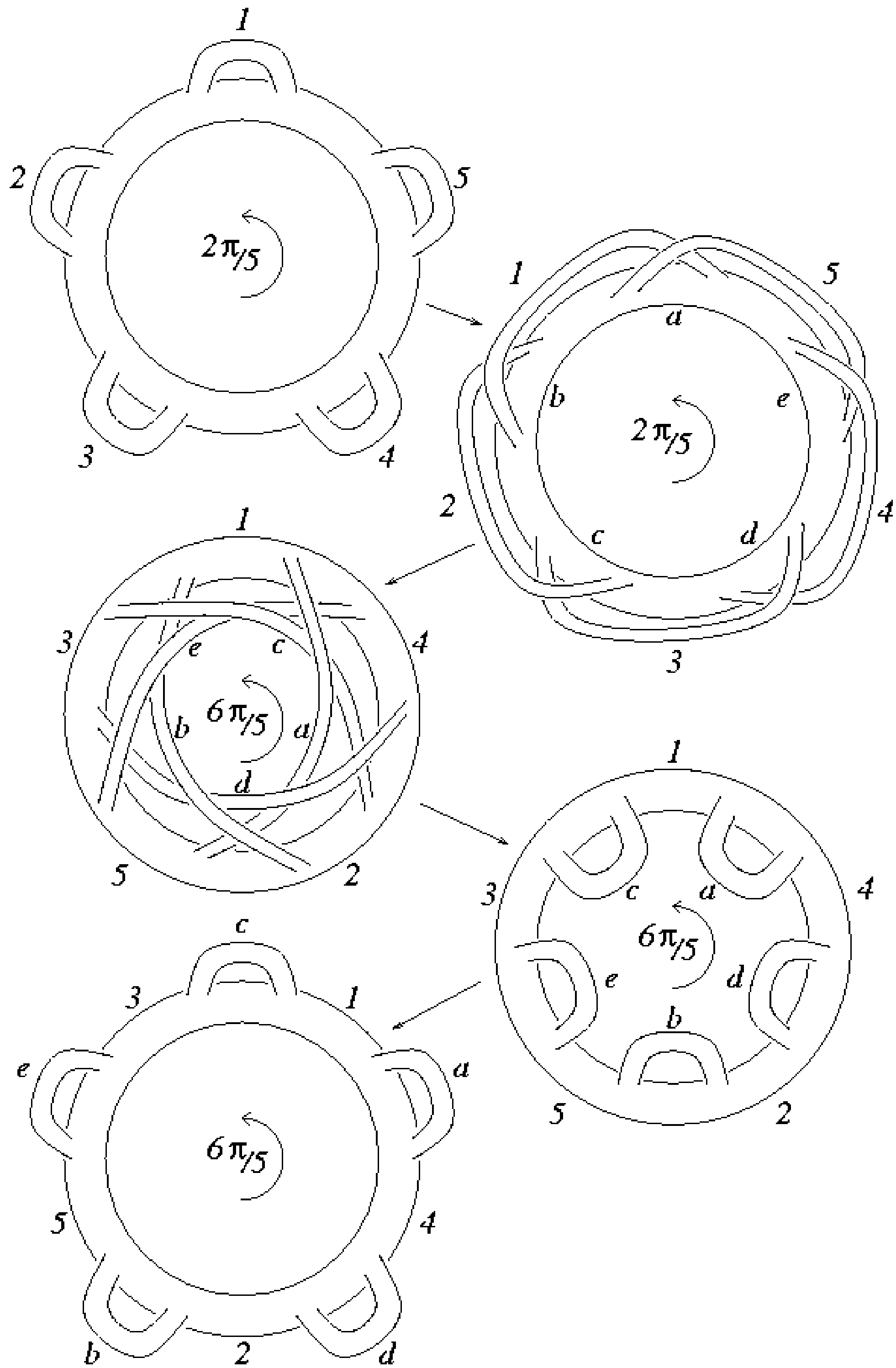
(That is, there exist a homeomorphism h of V and an automorphism α of G so that $\phi(\alpha(g)) = h \circ \psi(g) \circ h^{-1}$ for all $g \in G$.)

Example: For $G = C_5 = \{1\,, t\,, t^2, t^3, t^4\}$, define actions ϕ and ψ on the solid torus $V_1 = S^1 \times D^2$ by:

$$\phi(t)(\theta, x) = (e^{2\pi i/5}\theta, x)$$
$$\psi(t)(\theta, x) = (e^{6\pi i/5}\theta, x)$$

These are weakly equivalent, since if $\alpha(t) = t^3$ then $\phi(\alpha(t)) = \psi(t)$, but are not equivalent (using a result we will state later). However, *after a single stabilization, they become equivalent.*

Geometrically, this is complicated. The next page is a sequence of pictures showing the steps in constructing an equivalence of the stabilized actions:



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Although the determination of when two actions are equivalent is geometrically complicated, there is a simple group-theoretic criterion one can use to test equivalence and weak equivalence.

This criterion for equivalence was known to Kalliongis & Miller a number of years ago, in fact it appears between the lines of some of their published work, and was probably known to others as well.

The criterion uses a classical concept in group theory, called *Nielsen equivalence* of generating sets of G. It was studied by J. Nielsen, J. Thompson, B. & H. Neumann, and others.

Nielsen equivalence for generating sets of $\pi_1(M^3)$ has been used by Y. Moriah and M. Lustig to detect nonisotopic Heegaard splittings of various kinds of 3-manifolds.

Define a *generating* n*-vector* for G to be a vector $(g_1, \ldots, g_n)$, where $\{g_1, \ldots, g_n\}$ generates G. Two generating n-vectors $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_n)$ are related by an *elementary Nielsen move* if $(h_1, \ldots, h_n)$ equals one of:

1. $(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$ for some permutation σ,
2. $(g_1, \ldots, g_i^{-1}, \ldots, g_n)$,
3. $(g_1, \ldots, g_i g_j^{\pm 1}, \ldots, g_n)$, where $j \neq i$,

Call $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ *Nielsen equivalent* if they are related by a sequence of elementary Nielsen moves, and *weakly Nielsen equivalent* if $(\alpha(s_1), \ldots, \alpha(s_n))$ and $(t_1, \ldots, t_n)$ are Nielsen equivalent for some automorphism α of G.

Using only elementary covering space theory, one can check that:

The (weak) equivalence classes of free $\mathbf{G}$-actions on $\mathbf{V_{1+(n-1)|G|}}$ correspond to the (weak) Nielsen equivalence classes of generating $\mathbf{n}$-vectors of $\mathbf{G}$.

Example revisited: For $G = C_5 = \{1\,, t\,, t^2, t^3, t^4\}$, define actions ϕ and ψ on the solid torus $V_1 = S^1 \times D^2$ by:

$$\phi(t)(\theta, x) = (e^{2\pi i/5}\theta, x)$$
$$\psi(t)(\theta, x) = (e^{6\pi i/5}\theta, x)$$

These actions are inequivalent, but after one stabilization, they become equivalent:

Proof: (t) is not Nielsen equivalent to (t^3), but

$(t, 1) \sim (t, t^3) \sim (tt^{-3}t^{-3}, t^3) = (1, t^3) \sim (t^3, 1)$ $\Box$

Notation: Fix G. For $k \geq 0$, define

$e(k) =$ the number of equivalence classes of G-actions on $V_{1+(\mu+k-1)|G|}$,

$w(k) =$ the number of weak equivalence classes of G-actions on $V_{1+(\mu+k-1)|G|}$.

Note that

1. For all k, $1 \leq w(k) \leq e(k)$.

2. $w(0)$ is the number of weak equivalence classes of minimal genus free G-actions.

3. $e(k) = 1$ for all $k \geq 1$ means that any two free G-actions on a handlebody of genus above the minimal genus are equivalent.

Some results, mostly proven by quoting good algebra done by other people.

1. (B. & H. Neumann) For $G = A_5$, $w(0) = 2$. That is, there are two weak equivalence classes of A_5-actions on V_{61}.

2. (D. Stork) For $G = A_6$, $w(0) = 4$. That is, there are four weak equivalence classes of A_6-actions on V_{361}.

3. (M. Dunwoody) For G solvable:
   $w(0)$ can be arbitrarily large
   $e(k) = 1$ for all $k \geq 1$

4. (elementary) For G abelian, say $G = C_{d_1} \times \cdots \times C_{d_m}$ where $d_{i+1} | d_i$:

$$w(0) = 1$$
$$e(0) = \begin{cases} 1 & \text{if } d_m = 2 \\ \phi(d_m)/2 & \text{if } d_m > 2 \end{cases}$$

   A similar result holds for G dihedral.

5. (easy algebra) [various results saying that actions become equivalent after enough stabilizations]

6. (R. Gilman) For $G = \mathrm{PSL}(2,p)$, p prime, $e(k) = 1$ for $k \geq 1$. This includes the case of $\mathrm{PSL}(2,5) \cong A_5$.

7. (M. Evans) For $G = \mathrm{PSL}(2,2^m)$ or $G = \mathrm{Sz}(2^{2m-1})$, $e(k) = 1$ for $k \geq 1$.

8. (harder work using information about the subgroups of $\mathrm{PSL}(2,q)$, together with ideas of Gilman and Evans) For $G = \mathrm{PSL}(2,3^p)$, p prime, $e(k) = 1$ for $k \geq 1$. This includes the case of $\mathrm{PSL}(2,9) \cong A_6$. The same can probably be proven for more cases of $\mathrm{PSL}(2,q)$ using these methods.

Simple but difficult questions:

1. Are all actions on genera above the minimal one equivalent?

   I. e. is $e(k) = 1$ for all $k \geq 1$ for *all* finite G?

   I. e. if $n > \mu$, are any two generating n-vectors Nielsen equivalent?
   (For some *infinite* G, no)

2. Is every action the stabilization of a minimal genus action?
   I. e. is every generating n-vector equivalent to one of the form $(g_1, \ldots, g_\mu, 1, \ldots, 1)$?

3. Do any two G-actions on a handlebody become equivalent after one stabilization?

Yes for 1 $\Longleftrightarrow$ Yes for both 2 and 3.

A question that is probably much easier:

> Do there exist weakly inequivalent actions of a nilpotent G on a handlebody of genus less than 8193?

(This is the lowest-genus example we have found of inequivalent actions of a nilpotent group, it is a certain 3-generator nilpotent group. An example was given many years ago by B. H. Neumann, a 2-generator nilpotent group acting on the same genus.)