Primal Scream

Here is an intriguing elementary problem that I heard from Noel Brady:

> I pick two numbers between 2 and 100.
> I give their product to Pierre.
> I give their sum to Serge.
> Pierre says: "I do not know the numbers."
> Serge says: "I knew that."
> Pierre says: "Ah, now I know the numbers."
> Serge says: "Okay, so do I."
>
> What are the two numbers?

The first sentence on the next page tells the two numbers, and is followed by the proof that they really are the solution. The rest of the current page is blank, in order not to give away the answer for the reader who would like to give this problem a try.

The numbers are 4 and 13, but it will take some effort to see why. To get started, we will make some definitions that will enable us to restate the problem more mathematically.

1. A factorization $P = x \cdot y$ is *admissible* if $2 \leq x, y \leq 100$.
2. A number is *primal* if it has a unique admissible factorization.
3. A number S is the *weight* of an admissible factorization $P = x \cdot y$ when $S = x + y$. If P is primal, the weight of its unique admissible factorization is called the *weight of* P.
4. A number S is *good* if $S \leq 200$ and S is not the weight of any primal number.
5. A factorization $P = x \cdot y$ is *good* if its weight is a good number.
6. A number P is *distinguishable* if it has exactly one good factorization, in which case the weight of the good factorization is said to *distinguish* P.

In the original problem, Pierre is given P, and states that it is not primal. Serge states that his number S is good. Since Pierre is then able to determine the factorization of P, it must be that P is distinguishable. Since Serge can then deduce the numbers, S must distinguish only P. The wording of the original problem implies that the choice for x and y is actually unique. So the original problem can be restated as follows:

> Prove that there exists a unique pair of numbers x and y with $2 \leq x, y \leq 100$, so that if $P = xy$ and $S = x + y$, the following hold:
>
> 1. P is not primal.
> 2. S is good.
> 3. P is distinguishable.
> 4. S distinguishes only one number.

In the remainder of this note, we will prove that $\{x, y\} = \{4, 13\}$ is the only solution.

Data:

1. Every integer greater than 1 factors uniquely into a product of primes.
2. The primes less than 55 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, and 53.
3. The number 97 is prime.

Examples. In these examples, x and y always denote numbers with $2 \leq x, y \leq 100$, and p and q denote distinct primes with $p < q$.

1. A product of two primes is primal. If $p > 50$, any product xp is primal. For $p < 10$, p^3 is primal. Products of the form pq^2 are primal when $q^2 > 100$. The numbers 5^4, 7^4, 3^7, 3^8, 2^{11}, and 2^{12} are primal and have weight at least 50.
2. No number greater than 54 is good. For if $55 \leq S \leq 153$, then S can be written as $x + 53$, and $x \cdot 53$ is primal. Similarly, if $99 \leq S \leq 197$, then $S = x + 97$. If $198 \leq S \leq 200$, then $S = x + y$ with $99 \leq x, y \leq 100$, and $x \cdot y$ is primal.
3. Any even number less than 55 is a sum of two primes less than 100, so is not good.
4. No number of the form $2 + p$ is good, since $2 \cdot p$ is primal.

As a consequence of examples 2, 3 and 4, the only possible good numbers are the odd numbers less than 55 that are not of the form $2 + q$ with q prime. Also, 51 is not good, since $2 \cdot 17 \cdot 17$ is primal with weight 51. This leaves just ten possibilities, and the next proposition says that they are indeed good.

Proposition 1. *The good numbers are* 11*,* 17*,* 23*,* 27*,* 29*,* 35*,* 37*,* 41*,* 47*, and* 53*.*

For the proof, we need to know the even primal numbers of weight less than 54.

Lemma 2. *An even primal number of weight less than* 54 *is either a product of two primes, or is one of* $2 \cdot 2 \cdot 2$*,* $2 \cdot 11 \cdot 11$*,* $2 \cdot 13 \cdot 13$*, or* $2 \cdot 17 \cdot 17$*.*

Proof. The admissible factorization of such a number P may be written as $2z \cdot wq$, with q prime and $q < 50$. Since $2z + wq < 54$, we have $z \leq 25$ and $w \leq 25$.

If $z = w = 1$, then P is a product of two primes. If $z = 1$ and $w > 1$, then $2w \cdot q$ is an admissible factorization. Since it must be the same as $2 \cdot wq$, we have $q = 2$. So $2q \cdot w$ is also an admissible factorization, and $w = 2$ as well, giving the case $P = 2 \cdot 2 \cdot 2$. So we may assume that $z > 1$. Then, $wq < 50$ so $z \cdot 2wq$ is an admissible factorization. Therefore $z = wq$ and $P = 2wq \cdot wq$. Since $3wq < 54$, we have $w \leq 8$. Suppose for contradiction that $w > 1$. Since $3wq < 54$, we then have $q \leq 7$, so $2w^2 \cdot q^2$ is an admissible factorization. Therefore $2w^2$ is either $2wq$ or wq, and q is either w or $2w$. The product $P = 2wq \cdot wq$ must be either $2q^2 \cdot q^2$ or $4w^2 \cdot 2w^2$. Since the weight is less than 54, $P = 2q^2 \cdot q^2$ implies that $q^2 < 17$ so $q = 2$ or $q = 3$, but for these values of q, $2q^4$ is not primal. If $P = 4w^2 \cdot w^2$, then $5w^2 < 54$ so $w = 2$ or $w = 3$, and $P = 4w^2 \cdot w^2$ is not primal. We conclude that $w = 1$, so P has the form $2q^2$. One possibility is $q = 2$, the first of the four special cases listed in the theorem. If $q > 2$, then $11 \leq q$, since otherwise $2q \cdot q$ and $2 \cdot q^2$ would be distinct admissible factorizations. Also $q \leq 17$, since if not, then the weight $3q$ is greater than 54. So we have only the last three possibilities listed in the proposition. □

Proof of proposition 1. We saw in the Examples that no numbers other than the listed ones are good. None of the listed numbers can be the weight of an odd primal number, since their weights are even. Lemma 2 shows that the only possible weights less than 50 for even primals are either sums of two primes, or are the numbers 6, 33, 39, or 51. So all numbers on the list are good. □

Now, we will use the products $P = 2 \cdot 2 \cdot p$ and $P = 2 \cdot p \cdot q$ where p and q are primes with $2 < p < q < 50$. The first kind always have exactly two admissible factorizations. The second have exactly two admissible factorizations, when $pq > 100$, and exactly three when $pq < 100$. For each good number S except 17, Table 1 gives two such numbers P each having exactly one good factorization, with the weight of that factorization being S. That is, all of these other values of S distinguish at least two numbers. In the table, E stands for an even number less than 55, and L for a number larger than 54.

At this point, we know that the only possibility for a good number that distinguishes only one number is $S = 17$. It remains to see that there is only number distinguished by 17, that is, only one choice for P having a factorization with weight 17, for which that is the only good factorization of P. The choice is $P = 52$. Its good factorization $4 \cdot 13$ has weight 17, while its other factorization $2 \cdot 26$ has weight 28 so is not good, Table 2 shows two good factorizations for each other P having a factorization of weight 17, thereby eliminating all other possibilities for P.

We can now tell the story of Pierre and Serge with the actual numbers. Pierre was given $P = 52$, which might be either $2 \cdot 26$ or $4 \cdot 13$. Serge was given $S = 17$, and announces that his number is good. Knowing that the weight of the factorization $x \cdot y$ is good, Pierre can then eliminate the possibility of $2 \cdot 26$, and then knows x and y. Since 52 is the only number having a factorization of weight 17 that is good, but having no other good factorization, Serge then knows that x and y must have been 4 and 13.

S	P	weights
11	$2 \cdot 2 \cdot 7$	$11\,,\,E$
	$2 \cdot 3 \cdot 3$	$11\,,\,9$
23	$2 \cdot 2 \cdot 19$	$23\,,\,E$
	$2 \cdot 5 \cdot 13$	$23\,,\,31\,,\,L$
27	$2 \cdot 2 \cdot 23$	$23\,,\,E$
	$2 \cdot 5 \cdot 11$	$27\,,\,21\,,\,L$
29	$2 \cdot 3 \cdot 23$	$29\,,\,49\,,\,L$
	$2 \cdot 5 \cdot 19$	$29\,,\,43\,,\,L$
35	$2 \cdot 2 \cdot 31$	$35\,,\,L$
	$2 \cdot 3 \cdot 29$	$35\,,\,L\,,\,L$
37	$2 \cdot 7 \cdot 23$	$37\,,\,L\,,\,L$
	$2 \cdot 3 \cdot 31$	$37\,,\,L\,,\,L$
41	$2 \cdot 2 \cdot 37$	$41\,,\,L$
	$2 \cdot 3 \cdot 19$	$41\,,\,25\,,\,L$
47	$2 \cdot 2 \cdot 43$	$47\,,\,L$
	$2 \cdot 3 \cdot 41$	$47\,,\,L\,,\,L$
53	$2 \cdot 3 \cdot 47$	$53\,,\,L\,,\,L$
	$2 \cdot 5 \cdot 43$	$53\,,\,L\,,\,L$

TABLE 1. The good numbers other than 17 distinguish more than one P.

$x+y$	P	two good factorizations
$2+15$	$2 \cdot 3 \cdot 5$	$2 \cdot 15\,,\ 6 \cdot 5$
$3+14$	$2 \cdot 3 \cdot 7$	$3 \cdot 14\,,\,2 \cdot 21$
$5+12$	$2 \cdot 2 \cdot 3 \cdot 5$	$5 \cdot 12\,,\,3 \cdot 20$
$6+11$	$2 \cdot 3 \cdot 11$	$6 \cdot 11\,,\,2 \cdot 33$
$7+10$	$2 \cdot 5 \cdot 7$	$7 \cdot 10\,,\,2 \cdot 35$
$8+\ 9$	$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$	$8 \cdot 9\,,\,3 \cdot 24$

TABLE 2. 17 distinguishes only the number 52.

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