Learning from my students

Darryl McCullough

University of Oklahoma

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I often say that I learn more from my students than they learn from me.

People think I'm joking.

So let me show you something they taught me this semester.

I am teaching Calculus IV— Multivariable Calculus for Engineers— for the thirteenth time.

Having taught it twelve times already, there isn't much I haven't seen. But on our first exam this semester, I learned something new.

I gave the following exam question, modeled on a problem from the book that had been assigned as a homework problem:

Let $x = e^u \sin(t)$, $y = e^u \cos(t)$, and $z = f(x, y)$.

1. Calculate $\dfrac{\partial x}{\partial t}$ and $\dfrac{\partial y}{\partial t}$.

$$\frac{\partial x}{\partial t} = e^u \cos(t) = y, \text{ and}$$

$$\frac{\partial y}{\partial t} = -e^u \sin(t) = -x.$$

2. Calculate $\dfrac{\partial z}{\partial t}$ and express it purely in terms of x, y, $\dfrac{\partial z}{\partial x}$, and $\dfrac{\partial z}{\partial y}$.

Applying the Chain Rule, we have

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = y\,\frac{\partial z}{\partial x} - x\,\frac{\partial z}{\partial y}\ .$$

3. Calculate $\dfrac{\partial}{\partial t}\left(x\,\dfrac{\partial z}{\partial x}\right)$, and express it purely in terms of in terms of x, y, and partial derivatives of z with respect to x and y.

Applying the Chain Rule, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(x\frac{\partial z}{\partial x}\right) &= \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial t} \\ &= \left(\frac{\partial z}{\partial x} + x\,\frac{\partial^2 z}{\partial x^2}\right)y + \left(x\,\frac{\partial^2 z}{\partial y\,\partial x}\right)(-x) \\ &= y\,\frac{\partial z}{\partial x} + xy\,\frac{\partial^2 z}{\partial x^2} - x^2\,\frac{\partial^2 z}{\partial y\,\partial x}\ .\end{aligned}$$

But several of the students came up with an approach I had never seen before:

$$\begin{aligned}\frac{\partial}{\partial t}\left(x\,\frac{\partial z}{\partial x}\right) &= \frac{\partial x}{\partial t}\,\frac{\partial z}{\partial x} + x\,\frac{\partial^2 z}{\partial t\,\partial x}\\ &= y\,\frac{\partial z}{\partial x} + x\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)\\ &= y\,\frac{\partial z}{\partial x} + x\,\frac{\partial}{\partial x}\left(y\,\frac{\partial z}{\partial x} - x\,\frac{\partial z}{\partial y}\right)\\ &= y\,\frac{\partial z}{\partial x} + x\left(y\,\frac{\partial^2 z}{\partial x^2} - x\,\frac{\partial^2 z}{\partial x\,\partial y} - \frac{\partial z}{\partial y}\right)\\ &= y\,\frac{\partial z}{\partial x} + xy\,\frac{\partial^2 z}{\partial x^2} - x^2\,\frac{\partial^2 z}{\partial x\,\partial y} - x\,\frac{\partial z}{\partial y}\ .\end{aligned}$$

The answer does not agree with our previous one:

$$y\,\frac{\partial z}{\partial x} + xy\,\frac{\partial^2 z}{\partial x^2} - x^2\,\frac{\partial^2 z}{\partial x\,\partial y}\ .$$

Of course the calculation using the Chain Rule must be correct. The error in the second is a faulty application of Clairaut's Theorem in this step:

$$\frac{\partial}{\partial t}\left(x\,\frac{\partial z}{\partial x}\right) = \frac{\partial x}{\partial t}\,\frac{\partial z}{\partial x} + x\,\frac{\partial^2 z}{\partial t\,\partial x}$$
$$= y\,\frac{\partial z}{\partial x} + x\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)$$

Clairaut's Theorem, which I had explained as "You can take partial derivatives in any order and get the same answer," does not apply, because x and t are not independent variables.

Here is a simple example from 1-variable calculus that shows that such a calculation does not work:

$$\frac{d}{dt}\frac{d}{d(t^2)}(t^2) = \frac{d}{dt}(1) = 0$$

but taking the derivatives in the other order gives

$$\frac{d}{d(t^2)}\frac{d}{dt}(t^2) = \frac{d}{d(t^2)}(2t)$$
$$= \frac{d}{d(t^2)}\left(2\sqrt{t^2}\right) = 2 \cdot \frac{1}{2\sqrt{t^2}} = \frac{1}{t}$$

Derivatives with respect to non-independent variables simply need not commute.