MORSE THEORY AND CONJUGACY CLASSES OF FINITE SUBGROUPS

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Abstract. We construct a CAT(0) group containing a finitely presented subgroup with infinitely many conjugacy classes of finite-order elements. Unlike previous examples (which were based on right-angled Artin groups) our ambient CAT(0) group does not contain any rank 3 free abelian subgroups.

We also construct examples of groups of type F_n inside mapping class groups, $\mathrm{Aut}(\mathbb{F}_k)$, and $\mathrm{Out}(\mathbb{F}_k)$ which have infinitely many conjugacy classes of finite-order elements.

Hyperbolic groups contain only finitely many conjugacy classes of finite subgroups (see [2, 5, 7]). Several other classes of groups share this property, including CAT(0) groups [7, Corollary II.2.8], mapping class groups [6], $\mathrm{Aut}(\mathbb{F}_k)$, $\mathrm{Out}(\mathbb{F}_k)$ [8], and arithmetic groups [3]. Building on work of Grunewald and Platonov [11], Bridson [6] showed that for any n, there is a subgroup of $SL(2n+2,\mathbb{Z})$ of type F_n that has infinitely many conjugacy classes of elements of order 4. In [10], Feighn and Mess constructed finite extensions of $(\mathbb{F}_2)^n$ containing subgroups of type F_{n-1} with infinitely many conjugacy classes of elements of order 2. Their examples were realized as subgroups of the isometry group of $(\mathbb{H}^2)^n = \mathbb{H}^2 \times \cdots \times \mathbb{H}^2$. These examples were generalized considerably and were set in the context of right-angled Artin groups by Leary and Nucinkis [12].

In this note we describe a model situation where the Morse theory argument of [12] can be applied. It includes the right-angled Artin group setting from [12], but it can also be used when the ambient group is not an Artin group. We apply it to the case of a hyperbolic group in Theorem 2.1 and to the case of a virtual direct product of hyperbolic groups in Theorem 2.2. In addition, we produce subgroups of mapping class groups, $\mathrm{Aut}(\mathbb{F}_k)$, and $\mathrm{Out}(\mathbb{F}_k)$ with infinitely many conjugacy classes of finite-order elements by finding natural realizations of finite extensions of $(\mathbb{F}_2)^n$ in these groups (Theorems 3.1, 3.2, and 3.3). For mapping class groups this solves Problem 3.10 in [9].

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1. Counting conjugacy classes of finite-order elements in subgroups of $\mathbb{Z}$-extensions

Proposition 1.1 below gives a basic method for producing examples of groups with infinitely many conjugacy classes of finite-order elements. The proof is contained in that of Theorem 3 in [12], where results are stated for right-angled Artin groups. As background for the model situation and Proposition 1.1, one needs the notion of a Morse function on an affine cell complex, and the notion of non-positively curved cell complexes. For details about the first topic the reader can refer to [1]. A good treatment of the latter topic can be found in [7].

Model Situation. Let X be a non-positively curved cell complex and let $f : X \to S^1$ be a circle-valued Morse function which induces an epimorphism of fundamental groups

$$f_* : G = \pi_1(X) \to \mathbb{Z} = \pi_1(S^1)$$

Let K denote the kernel of this epimorphism.

Now let σ be a finite order isometry of X which

- fixes a vertex $v \in X$, and
- acts without fixed points on the link $Lk(v, X)$.

Further, assume that the map f is σ-equivariant (for the trivial action of σ on S^1). The isometry σ induces an automorphism of G, which we also denote by σ. Then f_* is σ-equivariant, and it follows that σ leaves K invariant. We form the semi-direct product $G \rtimes \langle\sigma\rangle$, and extend f_* to this group by mapping σ trivially. Note that $G \rtimes \langle\sigma\rangle$ can be expressed as a semi-direct product with $\mathbb{Z}$

$$G \rtimes \langle\sigma\rangle \ = \ (K \rtimes \langle\sigma\rangle) \rtimes \mathbb{Z}.$$

Proposition 1.1. *Let σ, f, X, G and K be as described in the model situation above. Then the group $K \rtimes \langle\sigma\rangle$ has infinitely many conjugacy classes of elements with the same order as σ. In fact, the conjugacy class of σ in $G \rtimes \langle\sigma\rangle$ intersects $K \rtimes \langle\sigma\rangle$ to give infinitely many $K \rtimes \langle\sigma\rangle$ conjugacy classes.*

Proof. Let $t \in G$ be such that $f_*(t) = 1$. Then $t^n\sigma t^{-n}$ is an element of $K \rtimes \langle\sigma\rangle$. We show that $t^n\sigma t^{-n}$ is not conjugate to $t^m\sigma t^{-m}$ in $K \rtimes \langle\sigma\rangle$ unless $n = m$.

Since X is non-positively curved, its universal cover $\widetilde{X}$ is a CAT(0) space, and hence has unique geodesics [7]. Choose a lift $\tilde{\sigma} : \widetilde{X} \to \widetilde{X}$ which fixes a vertex $x_0 \in \widetilde{X}$ in the preimage of the vertex $v \in X$ fixed by σ. Thus $\tilde{\sigma}$ is an isometry of $\widetilde{X}$ which fixes the vertex x_0.

We argue by contradiction that this is the only point of $\widetilde{X}$ which is fixed by $\tilde{\sigma}$. If $\tilde{\sigma}$ fixed another point $x_1 \in \widetilde{X}$, then it would have to fix the unique geodesic $[x_0 x_1]$ (because $\tilde{\sigma}$ is an isometry). Hence $\tilde{\sigma}$ would fix the point of $Lk(x_0, \widetilde{X})$ determined by $[x_0 x_1]$. Since $\tilde{\sigma}$ is a lift of σ, this would imply that σ fixes a point of $Lk(v, X)$, contradicting the hypothesis on σ. Thus x_0 is the unique fixed point of $\tilde{\sigma}$.

Let $\tilde{f} : \widetilde{X} \to \mathbb{R}$ be a lift of f; it is a Morse function on $\widetilde{X}$. Since f is σ-equivariant where σ is defined to act trivially on S^1, the isometry $\tilde{\sigma}$ acts on $\widetilde{X}$ preserving height.

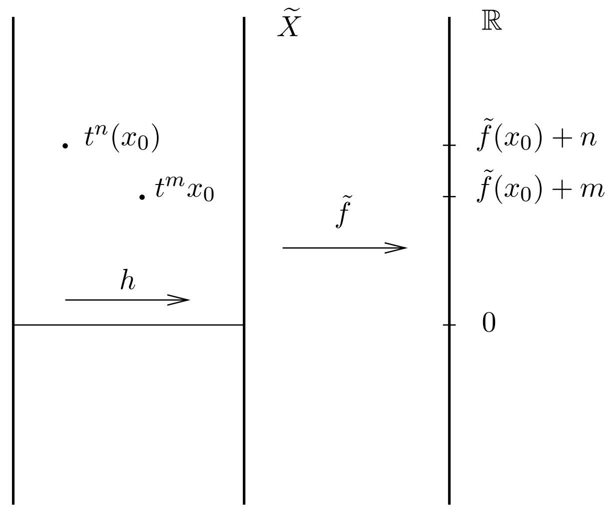

FIGURE 1. Picture proof of Proposition 1.1.

Recall that we chose $t \in G$ so that $f_*(t) = 1$. The element t acts as a deck transformation on $\widetilde{X}$, and so $t^k \tilde{\sigma} t^{-k}$ is a lift of σ with unique fixed point $t^k(x_0)$. Note that

$$\tilde{f}(t^k(x_0)) = \tilde{f}(x_0) + k\,.$$

However, each $h \in K \rtimes \langle \sigma \rangle$ acts *horizontally* on $\widetilde{X}$; that is, $\tilde{f}(h(x)) = \tilde{f}(x)$ for $x \in \widetilde{X}$. In particular h cannot move the point $t^m x_0$ to the point $t^n x_0$ for $m \neq n$. Therefore $t^n \sigma t^{-n} \neq h(t^m \sigma t^{-m}) h^{-1}$ for $m \neq n$, and so the elements $t^n \sigma t^{-n}$ each represent distinct conjugacy classes in $K \rtimes \langle \sigma \rangle$. □

The following family of examples provides illustrations of Proposition 1.1 in all dimensions. In Section 3 we shall see how to embed these examples into various classical groups.

Example 1.2. Let X be the direct product of n copies of the wedge of two circles, $S^1 \vee S^1$. Then $G = \pi_1(X)$, is isomorphic to $(\mathbb{F}_2)^n$. Define $f : X \to S^1$ by mapping each of the $2n$ circles homeomorphically around S^1 and extending linearly over the higher skeleta. Thus f is a circle valued Morse function, which lifts to a Morse function $\tilde{f}$ on the universal cover $\widetilde{X}$. The cover $\widetilde{X}$ is tiled by cubes. On each cube the map $\tilde{f}$ attains a maximum value at one vertex, and attains a minimum at the diametrically opposite vertex.

Define an order 2 automorphism σ of X as follows. In the case $n = 1$, σ simply interchanges the two circles. In the case $n \geq 2$, extend this to a diagonal action. Note that σ fixes the unique vertex of X. The link of this vertex is the n-fold join of a set of four points. Since σ acts on a single set of four points by interchanging points in pairs, the action on the whole link has no fixed points. Clearly, f is σ-equivariant.

The complex X is non-positively curved since it is the n-fold direct product (with the product metric) of the non-positively curved 1-complex $S^1 \vee S^1$. If the two circles of $S^1 \vee S^1$ are chosen to be isometric (eg. both are $\mathbb{R}/\mathbb{Z}$) then the map σ which interchanges them can be taken to be an isometry. Similarly, the diagonal map σ acting on X can be taken to be an isometry. Thus all the hypotheses of Proposition 1.1 are satisfied.

By Proposition 1.1, the kernel of the map $G \rtimes \langle \sigma \rangle \to \mathbb{Z}$ has infinitely many conjugacy classes of elements of order 2. .

Note that these kernels are of type F_{n-1} but not of type F_n [1]. Examples similar to these were given in Feighn-Mess [10], and were generalized considerably by Leary-Nucinkis [12].

2. Conjugacy classes in subgroups of CAT(0) groups

In this section we give two applications of Proposition 1.1. In the first example, the fundamental group of the complex X is hyperbolic, and the kernel of the map to $\mathbb{Z}$ is finitely generated but not finitely presented. The group $G = \pi_1(X)$ is extended by a carefully chosen finite-order automorphsim.

Theorem 2.1. *There exist hyperbolic groups containing finitely generated subgroups which have infinitely many conjugacy classes of finite order elements.*

For the second example, we take the direct product of two copies of the complex X of the first example, and take the diagonal action of the finite-order automorphism. As in the case of the direct product of free groups (in Example 1.2 above) the finiteness properties of the kernel improve. In this case the kernel is of type F_3 but not F_4. The ambient CAT(0) group has $\mathbb{Z}^2$ subgroups, but not $\mathbb{Z}^3$ subgroups.

Theorem 2.2. *There exist CAT(0) groups with no $\mathbb{Z}^3$ subgroups, which contain finitely presented (in fact type F_3) subgroups which have infinitely many conjugacy classes of finite order elements.*

Remark 2.3. It would be very interesting to find a hyperbolic group which contains a finitely presented subgroup with infinitely many conjugacy classes of finite-order elements. The subgroup in question could not be a hyperbolic group.

The hyperbolic example. In this subsection we prove Theorem 2.1 by construction. The construction produces one example, but it should be clear how to vary the construction to obtain other examples.

Start with the group $G = \pi_1(X)$, where X is a 2-complex consisting of one vertex, eight 1-cells and eight hexagonal 2-cells as shown in Figure 2.

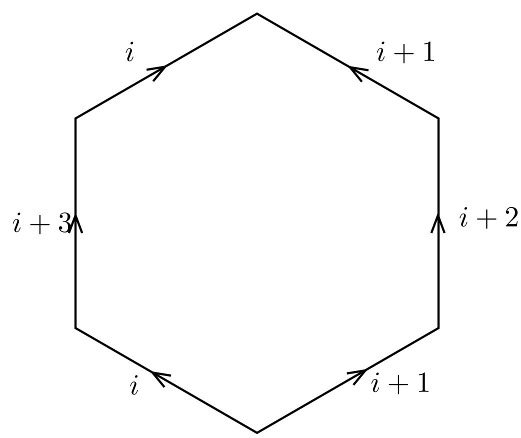

FIGURE 2. A typical 2-cell of X.

Define a circle-valued Morse function $f : X \to S^1$ by mapping each oriented 1-cell homeomorphically around S^1, and extending linearly over the 2-skeleton. In the universal cover $\widetilde{X}$, the Morse function $\widetilde{f}$ "projects" a typical 2-cell (as shown in Figure 2) horizontally onto a segment of length 3 in $\mathbb{R}$.

The ascending link is a circle with 8 vertices labelled i^-, $1 \leq i \leq 8$, and 1-cells from i^- to $(i+1)^-$ ($i \mod 8$). The descending link is described similarly, with i^+ in place of i^- above. The full link has the following additional edges: for each i, there are edges connecting the vertex i^+ to $(i \pm 1)^-$ and $(i \pm 3)^-$ (mod 8). Figure 3 shows how the vertex 1^+ of the descending link is connected to 4 vertices of the ascending link.

The complex X is given a piecewise hyperbolic metric by making every 2-cell a regular right-angled hyperbolic hexagon. From Figure 3, we see that there are no cycles of combinatorial length less than 4 (or metric length less than 2π). By the large link condition, X is a non-positively curved complex; the universal cover $\widetilde{X}$ is a CAT(-1) space, and so G is hyperbolic. See [7] for details about CAT(-1) spaces and link conditions.

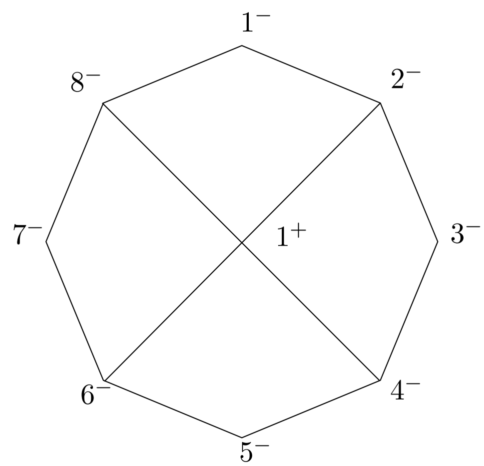

FIGURE 3. Part of the link of the vertex of X.

Define the finite-order automorphism σ of X to be the cellular homeomorphism of the 2-complex which fixes the vertex and cyclically permutes the oriented 1-cells (and hence also cyclically permutes the 2-cells). Note that f is σ-equivariant, once we define σ to act trivially on S^1. By construction, σ acts without fixed points (as an order 8 rotation) on the ascending and the descending links of the single vertex of X. The remaining edges of the link connect $+$ vertices to $-$ vertices. These are freely permuted by σ since σ freely permutes $+$ vertices (and freely permutes $-$ vertices). Finally, σ can easily be taken to be an isometry of X. Thus all the hypotheses of Proposition 1.1 are satisfied.

The fact that the ascending and descending links are each isomorphic to S^1 implies that the kernel K of the map $G \to \mathbb{Z}$ is finitely generated but not finitely presented by Theorem 4.7(1) of [4] . Now the finite extension $G \rtimes \langle\sigma\rangle$ is virtually G, and hence is also hyperbolic. The kernel $K \rtimes \langle\sigma\rangle$ is virtually K, and so is also fintely generated but not finitely presented. By Proposition 1.1 $K \rtimes \langle\sigma\rangle$ has infinitely many conjugacy classes of elements of order 8. □

Increasing the finiteness properties of the kernel. In this subsection, we give a proof of Theorem 2.2 by construction.

Start with the hyperbolic group G, the 2-complex X, and the Morse function $f : X \to S^1$ of the preceding example.

The group $G \times G$ is the fundamental group of the product space $X \times X$. Consider the composition of the product map

$$f \times f \;:\; X \times X \;\to\; S^1 \times S^1$$

with the standard "linear map" $S^1 \times S^1 \to S^1$ (covered by the linear map $\mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x + y$). This is a circle valued Morse function on $X \times X$, whose ascending (resp. descending) link is simply a join of two copies of the ascending (resp. descending) link of f. Topologically, these ascending and descending links are 3-spheres. Hence the kernel K of the induced map $G \times G \to \mathbb{Z}$ is of type F_3 but not of type F_4 (or even FP_4) by Theorem 4.7(1) of [4].

Since X is non-positively curved, the product complex $X \times X$ with the product metric is also non-positively curved. Hence, the universal cover $\widetilde{X} \times \widetilde{X}$ is a CAT(0) metric space. The group $G \times G$ is a CAT(0) group. Since it is the direct product of two hyperbolic groups, the maximal rank of its free abelian subgroups is 2.

The order eight isometry $\sigma : X \to X$ defines an isometry $X \times X \to X \times X$ (acting in a diagonal fashion) which we also denote by σ. The Morse function on $X \times X$ is σ-equivariant, once σ is defined to act trivially on S^1.

The isometry σ fixes the unique vertex of $X \times X$. We see that σ acts without fixed points on the link of this vertex as follows. First note that the ascending (resp. descending) link of the Morse function on $X \times X$ is a 3-sphere, expressed as the join of two circles, each of combinatorial size eight. Since σ acts as an order eight rotation on each circle, it freely permutes all the simplices of these 3-spheres. Finally, since a general simplex of the link will be a join of a simplex of the ascending link and a simplex of the descending link, we see that σ freely permutes all simplices of the link.

Proposition 1.1 implies that the kernel $K \rtimes \langle\sigma\rangle$ of the induced map

$$(G \times G) \rtimes \langle\sigma\rangle \to \mathbb{Z}$$

has infinitely many conjugacy classes of elements of order eight.

Finally, note that the group $(G \times G) \rtimes \langle\sigma\rangle$ is a CAT(0) group, because σ is a finite order isometry of $X \times X$. Also, the kernel $K \rtimes \langle\sigma\rangle$ is virtually K, and so shares the same finiteness properties (F_3 but not F_4). □

3. Mapping class groups, Aut($\mathbb{F}_k$), and Out($\mathbb{F}_k$)

In this section we show that the group $(\mathbb{F}_2)^n \rtimes \langle\sigma\rangle$ from Example 1.2 can be embedded in the mapping class group of a surface of sufficiently high genus (this solves Problem 3.10 in [9]), in $\mathrm{Aut}(\mathbb{F}_k)$, and $\mathrm{Out}(\mathbb{F}_k)$, where the free group has sufficiently high rank.

Let Σ_g be a closed orientable surface of genus $g \geq 3$ and $\mathcal{MCG}(\Sigma_g)$ the mapping class group of Σ_g.

Theorem 3.1. *There exists a subgroup of type* F_{g-3} *in* $\mathcal{MCG}(\Sigma_g)$ *that contains infinitely many conjugacy classes of elements of order 2.*

To ensure that the example is finitely presented, we need $g \geq 5$.

Proof. Let a_i, b_i, $i = 1, \ldots, g-2$ be the simple closed curves pictured in Figure 4 and let T_{a_i}, T_{b_i} denote the Dehn twists about these curves. The only intersections between these curves are between a_i and b_i, which intersect twice. Hence by [13, Theorem 7], the subgroup of $\mathcal{MCG}(\Sigma_g)$ generated by T_{a_i}, T_{b_i} is isomorphic to $(\mathbb{F}_2)^{g-2}$.

Let σ be the hyperelliptic involution of Σ_g such that $\sigma(a_i) = b_i$, $\sigma(b_i) = a_i$ for all i. Then $\sigma T_{a_i} \sigma = T_{b_i}$, and hence the subgroup $H = \langle T_{a_i}, T_{b_i}, \sigma\rangle < \mathcal{MCG}(\Sigma_g)$ is isomorphic to $(\mathbb{F}_2)^{g-2} \rtimes \mathbb{Z}_2$. The kernel from Example 1.2 is the desired subgroup. ☐

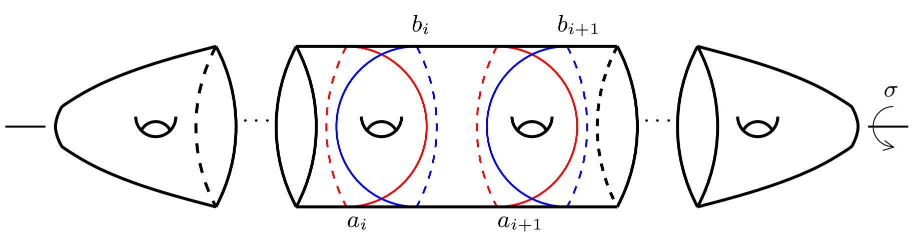

Figure 4. The curves from Theorem 3.1.

Letting Σ_g have punctures that are symmetric with respect to the hyperelliptic involution σ, we get examples of subgroups with infinitely many conjugacy classes of finite subgroups in $\mathrm{Out}(\mathbb{F}_k)$. The smallest finitely presented example occurs when $g = 3$ and there are 2 punctures, this provides an example in $\mathrm{Out}(\mathbb{F}_7)$. We give an alternative example below.

Theorem 3.2. *There exists a subgroup of type* $\mathrm{F}_{\lfloor k/2 \rfloor -1}$ *in* $\mathrm{Aut}(\mathbb{F}_k)$ *that contains infinitely many conjugacy classes of elements of order 2.*

To ensure that the example is finitely presented, we need $k \geq 6$. Note that $\lfloor k/2 \rfloor$ denotes the floor of $k/2$.

Proof. Fix a basis $x_1, \dots, x_k$ of $\mathbb{F}_k$. For $i = 1, \dots, \lfloor k/2 \rfloor$ define the following automorphisms:

$$\phi_i\colon \begin{array}{rcl} x_{2i-1} & \mapsto & x_{2i-1}^2 x_{2i} \\ x_{2i} & \mapsto & x_{2i-1}x_{2i} \\ x_j & \mapsto & x_j \text{ if } j \neq 2i-1, 2i \end{array} \qquad \psi_i\colon \begin{array}{rcl} x_{2i-1} & \mapsto & x_{2i}x_{2i-1} \\ x_{2i} & \mapsto & x_{2i}^2 x_{2i-1} \\ x_j & \mapsto & x_j \text{ if } j \neq 2i-1, 2i \end{array}$$

For $k = 2$, the image of ϕ_1, ψ_1 in $\mathrm{GL}(2,\mathbb{Z})$ are the matrices $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$. There exist integers $m, n \geq 1$ such that the group $\langle \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]^m, \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]^n \rangle$ is free. Letting N denote the larger of m, n we see that $\langle \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]^N, \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]^N \rangle$ is free. Hence, as free groups are Hopfian, the subgroup $\langle \phi_1^N, \psi_1^N \rangle < \mathrm{Aut}(\mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2$. As ϕ_i commutes with ϕ_j, ψ_j when $j \neq i$, we see that the group generated by the automorphisms ϕ_i^N, ψ_i^N is isomorphic to $(\mathbb{F}_2)^{\lfloor k/2 \rfloor}$.

As in the case for the mapping class group, we have an involution $\sigma \in \mathrm{Aut}(\mathbb{F}_k)$ defined by $\sigma(x_{2i-1}) = x_{2i}, \sigma(x_{2i}) = x_{2i-1}$ for $i = 1, \dots, \lfloor k/2 \rfloor$ and $\sigma(x_k) = x_k$ if k is odd. It can easily be checked that $\sigma\phi_i\sigma = \psi_i$ and hence the subgroup $H = \langle \phi_i^N, \psi_i^N, \sigma \rangle < \mathrm{Aut}(\mathbb{F}_k)$ is isomorphic to $(\mathbb{F}_2)^{\lfloor k/2 \rfloor} \rtimes \mathbb{Z}_2$. The kernel from Example 1.2 is the desired subgroup. $\square$

Theorem 3.3. *There exists a subgroup of type* $\mathrm{F}_{\lfloor k/2 \rfloor - 1}$ *in* $\mathrm{Out}(\mathbb{F}_k)$ *that contains infinitely many conjugacy classes of elements of order 2.*

To ensure that the example is finitely presented, we need $k \geq 6$.

Proof. We claim that the subgroup H from Theorem 3.2 does not contain any nontrivial inner automorphisms. This implies that the subgroup of $\mathrm{Out}(\mathbb{F}_k)$ generated by the images of ϕ^N, ψ^N, σ is isomorphic to H and hence the kernel from Example 1.2 is the desired subgroup.

Suppose some composition ρ of the $\phi_i^N, \psi_i^N, \sigma$ is an inner automorphism. Since the image of x_1 by any automorphism in H is a word in x_1, x_2, if $\rho(x_1) = xx_1x^{-1}$, then x is a word in x_1, x_2. Also, since the image of x_3 by any automorphism is a word in x_3, x_4, we see that if $\rho(x_3) = xx_3x^{-1}$ then x is a word in x_3, x_4. Therefore x is the identity and ρ is trivial. $\square$

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