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Honors Calculus I [1823-001] Midterm I

Monday, September 25, 2000

For full credit, give reasons for all your answers.

Q1]...[8 points] Evaluate the following limits, showing all your work. What is the geometrical significance of these limits (if there is any!)?

$$\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$$

We begin by rationalizing the numerator, using the expression for the difference of two squares $(a-b)(a+b)=a^2-b^2$.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x} &= \lim_{x\to 0}\left(\frac{\sqrt{2+x}-\sqrt{2}}{x}\right)\left(\frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}\right)\\
&= \lim_{x\to 0}\frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})}\\
&= \lim_{x\to 0}\frac{x}{x(\sqrt{2+x}+\sqrt{2})}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}} \qquad \text{since } x\neq 0\\
&= \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}
\end{aligned}$$

The geometric significance of this limit is that it is the slope of the tangent line to the graph of $y=\sqrt{x}$ at the point $(2,\sqrt{2})$.

$$\lim_{x\to 0}\frac{\sqrt[3]{2+x}-\sqrt[3]{2}}{x}$$

We begin by rationalizing the numerator, now using the expression for the difference of two cubes $(a-b)(a^2+ab+b^2)=a^3-b^3$.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt[3]{2+x}-\sqrt[3]{2}}{x} &= \lim_{x\to 0}\left(\frac{\sqrt[3]{2+x}-\sqrt[3]{2}}{x}\right)\left(\frac{(2+x)^{2/3}+(2+x)^{1/3}2^{1/3}+2^{2/3}}{(2+x)^{2/3}+(2+x)^{1/3}2^{1/3}+2^{2/3}}\right)\\
&= \lim_{x\to 0}\frac{2+x-2}{x((2+x)^{2/3}+(2+x)^{1/3}2^{1/3}+2^{2/3})}\\
&= \lim_{x\to 0}\frac{x}{x(((2+x)^{2/3}+(2+x)^{1/3}2^{1/3}+2^{2/3})}\\
&= \lim_{x\to 0}\frac{1}{(2+x)^{2/3}+(2+x)^{1/3}2^{1/3}+2^{2/3}} \qquad \text{since } x\neq 0\\
&= \frac{1}{2^{2/3}+2^{2/3}+2^{2/3}} = \frac{1}{3(2^{2/3})}
\end{aligned}$$

The geometric significance of this limit is that it is the slope of the tangent line to the graph of $y=\sqrt[3]{x}$ at the point $(2,\sqrt[3]{2})$.

Q2]...[15 points] Write down (no explanations necessary) the following limits that we encountered in class.

$$\lim_{x\to 0} \cos x = 1$$

$$\lim_{x\to 0} \sin x = 0$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Use the limits above to compute the values of the following limits.

$$\lim_{x\to 0} \tan x$$

$$\begin{aligned}\lim_{x\to 0} \tan x &= \lim_{x\to 0} \frac{\sin x}{\cos x} \\ &= \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} \cos x} \\ &= \frac{0}{1} = 0\end{aligned}$$

$$\lim_{x\to 0} \frac{\tan x}{x}$$

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x}{x} &= \lim_{x\to 0} \frac{\sin x}{x \cos x} \\ &= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1}{\cos x} \\ &= 1.\frac{1}{1} = 1\end{aligned}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}$$

Hint: multiply by $\frac{1+\cos x}{1+\cos x}$ and see what happens.

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos x}{x^2} &= \lim_{x\to 0} \left(\frac{1-\cos x}{x^2}\right)\left(\frac{1+\cos x}{1+\cos x}\right) \\ &= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} \\ &= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} \\ &= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \lim_{x\to 0} \frac{1}{(1+\cos x)} \\ &= 1^2 \frac{1}{1+1} = \frac{1}{2}\end{aligned}$$

Q3]...[7 points] State the Intermediate Value Theorem (IVT).

Suppose $f(x)$ is a continuous function which is defined on the interval $[a, b]$, and suppose that N is a number between $f(a)$ and $f(b)$. Then there is a number c between a and b such that $f(c) = N$.

Use the IVT to show that there is a real number input for which the functions

$$f(x) = x^3 - 4x^2 + 2x + 1$$

and

$$g(x) = 2x^2 - x^3 - 1$$

have the same output.

Note that f and g will have the same output if and only if the the difference

$$h(x) = f(x) - g(x) = 2x^3 - 6x^2 + 2x + 2$$

has a real root.

But h is continuous (cubic polynomial) and $h(0) = 2 > 0$, and h tends to minus infinity as x gets large and negative. In particular, $h(-1) = -8 < 0$ so the IVT tells us that there is a root between -1 and 0.

Note: you may have accidentally stumbled on another root (namely 1) in answering this question!

Q4]...[15 points] Say whether the following statements are true (T), or false (F). Support your answers by giving reasons or examples where appropriate.

- The sum of two rational numbers is a rational number.

  True. In fact $p/q = r/s = (ps + qr)/qs$ which is clearly rational.

- The sum of two irrational numbers is an irrational number.

  False. For example $\sqrt{2}$ and $-\sqrt{2}$ are two irrational numbers, yet their sum is the rational number 0. [Likewise you can try $\sqrt{2}$ and $1 - \sqrt{2}$ whose sum is the rational number 1]

- The sum of a rational and an irrational number is an irrational number.

  True. If $p/q + \alpha$ were rational, then we could write $p/q + \alpha = r/s$ and rearrange to get $\alpha = r/s - p/q = (rq - ps)/qs$ which is rational, thus contradicting the irrationality of α.

- If $\lim_{x\to 0} f(x)$ does not exist and $\lim_{x\to 0} g(x)$ does not exist, then $\lim_{x\to 0} f(x) + g(x)$ must not exist.

  False. We have seen in class that if $f(x)$ is defined to be 1 for $x \leq 0$ and to be 0 for $x > 0$, and $g(x)$ is defined to be 0 for $x \leq 0$ and to be 1 for $x > 0$, then neither $\lim_{x\to 0} f(x)$ not $\lim_{x\to 0} g(x)$ exists. However, $f(x) + g(x) = 1$ for all values of x and so $\lim_{x\to 0}(f(x) + g(x)) = \lim_{x\to 0} 1 = 1$.

- If $\lim_{x\to 0} f(x) = 7$ then there is a number $\delta > 0$ such that $6 < f(x) < 8$ whenever $0 < |x| < \delta$.

  True. This follows from the formal definition of a limit when we take the output tolerance ϵ to be the number 1.