1... **[15 points]** Compute the following derivatives.

Find y' in terms of x and y where

$$x^2 + 4y^2 = 9$$

Implicit Diff:

$$\frac{d}{dx}(x^2+4y^2) = \frac{d}{dx}9$$

$$2x + 8y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{2x}{8y}}$$

Find $f^{(20)}$ where

$$f(x) = 5x^{21} - 4x^{19} + \sin(7x+3)$$

$\frac{d}{dx}(7x+3)$

$$f'(x) = 5(21)x^{20} - 4(19)x^{18} + 7\cos(7x+3)$$

$$\vdots$$

$$f^{(20)}(x) = 5(21)(20)\cdots(3)(2)x^1 - 0 + 7^{20}\sin(7x+3)$$

$$f^{(20)}(x) = 5(21!)x + 7^{20}\sin(7x+3)$$

sin $\downarrow$ cos $\downarrow$ $-$sin $\downarrow$ $-$cos

Multiples of 4

(Trig Diff, Power Rule, Ch. Rule)

Find $g'(x)$ where

$$g(x) = \int_{\tan(x)}^{5} \sqrt{1+t^2}\,dt$$

$$g(x) = -\int_5^{\tan(x)} \sqrt{1+t^2}\,dt$$

$$g'(x) = -\sqrt{1+\tan^2(x)}\cdot\sec^2(x)$$

(Fund Thm + Chain Rule)

.. **[10 points]** Evaluate the following integrals (one indefinite and one definite).

$$\int x \sec^2(x^2+1)\, dx$$

Subst$^{\underline{n}}$: $\quad u = x^2+1 \qquad du = 2x\, dx$

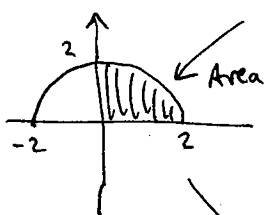
$$\frac{du}{2} = x\,dx$$

$$\int = \int \sec^2(u)\, \frac{du}{2} = \frac{1}{2}\tan(u) + C$$

$$= \frac{1}{2}\tan(x^2+1) + C$$

$$\int_0^2 (5-2x)\sqrt{4-x^2}\, dx$$

$$= 5\int_0^2 \sqrt{4-x^2}\, dx + \int_0^2 (-2x)\sqrt{4-x^2}\, dx$$



Subst$^{\underline{n}}$ $\quad u = 4-x^2$

$du = -2x\,dx$

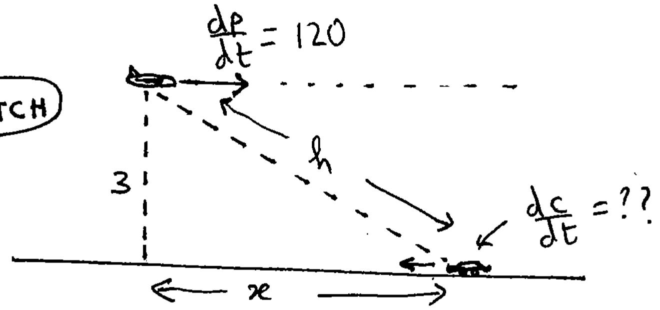
$$= 5\left(\frac{1}{4}\pi(2)^2\right) + \int_{4-0^2}^{4-2^2} u^{\frac{1}{2}}\, du$$

$$= 5\pi + \int_4^0 u^{\frac{1}{2}}\, du = 5\pi + \frac{2}{3}u^{3/2}\Big]_4^0$$

$$= 5\pi - \frac{16}{3}$$

Q3]. . . **[10 points]** A highway patrol plane flies 3 mi vertically above a level straight road. The plane flies horizontally (parallel to the road) at 120 mi/hr. The pilot sees an oncoming car on the road. The planes radar determines that at the instant the line-of-sight distance from the plane to the car is 5 mi, this distance is decreasing at a rate of 160 mi/hr. Determine the speed of the car along the highway at this instant.

SKETCH



h = line-of-sight distance

$$h^2 = x^2 + 3^2$$

$$\frac{d}{dt}$$

$$2h\frac{dh}{dt} = 2x\frac{dx}{dt} + 0$$

When $h = 5 \Rightarrow x^2 + 3^2 = 5^2$

$\Rightarrow x^2 = 16$

$x = 4$

and - - - - -

$$2(5)(-160) = 2(4)\frac{dx}{dt}$$

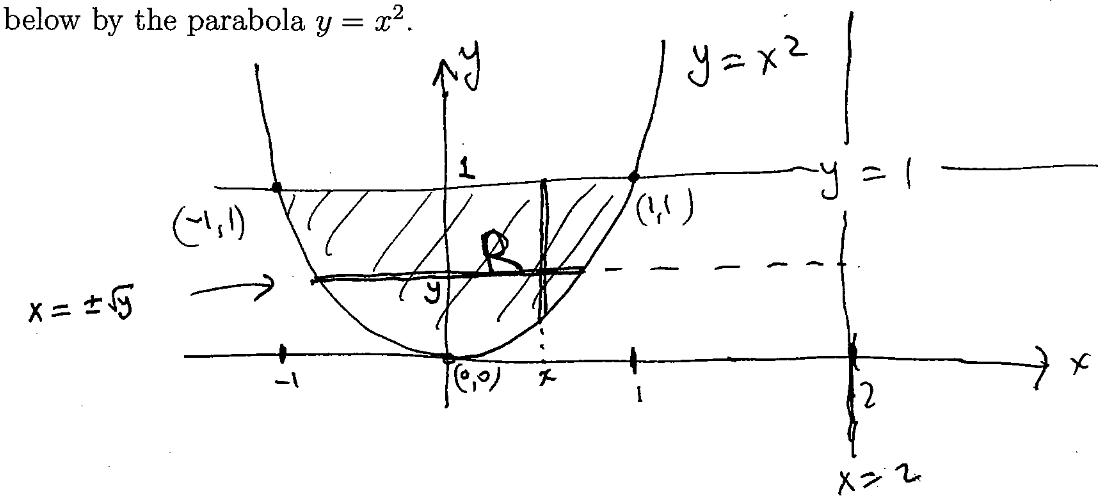
$$\frac{dx}{dt} = -\frac{5}{4}(160)$$

$$= -200 \text{ mi/hr}$$

$$-200 \text{ mi/hr} = -\left(\underbrace{\frac{dp}{dt}}_{120} + \frac{dc}{dt}\right)$$

$$\Rightarrow \frac{dc}{dt} = 200 - 120 = 80 \text{ mi/hr}$$

Q4]... **[10 points]** Draw the region R in the plane which is bounded above by the line $y = 1$ and bounded below by the parabola $y = x^2$.



Use the SHELL method to write down an integral for the volume obtained by rotating the region R about the line $x = 2$. You do not have to evaluate the integral.

$$\text{Vol} = \int_{-1}^{1} 2\pi\,(\text{radius})\,(\text{height})\,dx$$

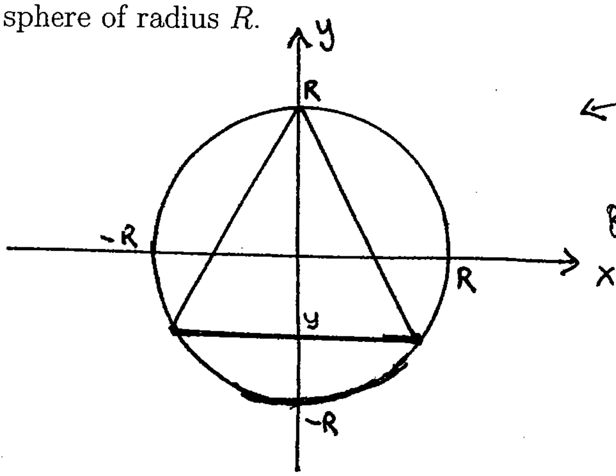
$$= \int_{-1}^{1} 2\pi\,(2-x)\,(1-x^2)\,dx$$

Use the WASHER (disk with hole) method to write down an integral for the volume obtained by rotating the region R about the line $x = 2$. You do not have to evaluate the integral.

$$\text{Vol} = \int_{0}^{1} \pi\left[(r_{out})^2 - (r_{inn})^2\right] dy$$

$$= \int_{0}^{1} \pi\left[(2+\sqrt{y})^2 - (2-\sqrt{y})^2\right] dy$$

Q5]...[10 points] Find the maximum volume of a right circular cone which can be inscribed inside of a sphere of radius R.



← CROSS SECTION. (Rotate about y-axis to get the 3-dimensional picture).

Base of Cone crosses y-axis at $y \Rightarrow$

$$(\text{height of cone}) = (R-y).$$

$$\left(\text{radius of cone's base}\right) = x = \sqrt{R^2-y^2}$$

$$\text{Vol Cone} = \frac{1}{3}\pi x^2 h$$

$$= \frac{1}{3}\pi (R^2-y^2)(R-y)$$

$$V(y) = \frac{\pi}{3}\left(R^3 - R^2 y - R y^2 + y^3\right)$$

$$\frac{dV}{dy} = 0 \Rightarrow \frac{\pi}{3}\left(0 - R^2 - 2Ry + 3y^2\right) = 0$$

$$3y^2 - 2Ry - R^2 = 0$$

$$(3y+R)(y-R) = 0$$

$$y = -\frac{R}{3} \qquad y = R \leftarrow (\text{Vol} = 0) \text{ not a max}$$

$$\text{Max vol} = \frac{\pi}{3}\left(R^2 - \left(\frac{-R}{3}\right)^2\right)\left(R - \left(\frac{-R}{3}\right)\right)$$

$$= \frac{\pi}{3}\left(\frac{8R^2}{9}\right)\left(\frac{4R}{3}\right) = \boxed{\frac{32\pi R^3}{81}}$$

Q6]... **[10 points]** Write down the formula for Newton's method of approximating roots of $f(x)$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

A classical algorithm for approximating the square root of a number A is the *divide and average* method. You have an estimate x_1 for $\sqrt{A}$. Now divide it into A. If the original estimate x_1 is larger (resp. smaller) than $\sqrt{A}$, then $\frac{A}{x_1}$ is smaller (resp. larger) than $\sqrt{A}$. So the average of the two

$$\frac{1}{2}\left(x_1 + \frac{A}{x_1}\right)$$

is taken to be the next approximation. Denote this by x_2 and repeat the process starting with x_2.

Show that the *divide and average* method is exactly the algorithm that Newton's method gives for approximating the root $\sqrt{A}$ of the function $f(x) = x^2 - A$.

$$f(x) = x^2 - A$$

$$f'(x) = 2x$$

Newton $\Rightarrow$ $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= x_1 - \frac{(x_1^2 - A)}{(2x_1)}$$

$$= x_1 - \frac{x_1}{2} + \frac{A}{2x_1}$$

$$= \frac{x_1}{2} + \frac{A}{2x_1}$$

$$= \frac{1}{2}\left(x_1 + \frac{A}{x_1}\right)$$

which is same expression!

Q7]...[15 points] Express the following limits as definite integrals.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i}{n}}$$

$$= \int_0^1 \sqrt{x}\, dx \Leftarrow \left[\begin{array}{l} x_i^* = \frac{i}{n} \\ i=0 \Rightarrow 0 \\ i=n \Rightarrow 1 \quad [0,1] \text{ interval.} \\ \Delta x = \frac{1-0}{n} = \frac{1}{n}. \end{array}\right.$$

$f(x) = \sqrt{x}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2}$$

$\div$ above + below by n^2

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\cdot\frac{1}{1+\left(\frac{i}{n}\right)^2} \longrightarrow \left[\begin{array}{l} x_i^* = \frac{i}{n} \\ \text{again } i=0 \Rightarrow 0 \\ i=n \Rightarrow 1 \\ \text{interval : } [0,1] \\ f(x) = \frac{1}{1+x^2} \\ \text{\& } \Delta x = \frac{1}{n} \end{array}\right.$$

$$= \int_0^1 \frac{dx}{1+x^2} \longleftarrow$$

If $f(x)$ is differentiable and $f'(2) = 7$, what is the value of the following limit?

$$\lim_{x\to 2} \frac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}}$$

$$= \lim_{x\to 2} \left(\frac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}} \cdot \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}} \right)$$

$$= \lim_{x\to 2} \left(\frac{f(x)-f(2)}{x-2} \right) \left(\sqrt{x}+\sqrt{2}\right)$$

$$= \lim_{x\to 2} \left(\frac{f(x)-f(2)}{x-2} \right) \cdot \lim_{x\to 2} \left(\sqrt{x}+\sqrt{2}\right)$$

$$= f'(2)\left(\sqrt{2}+\sqrt{2}\right) = 7(2\sqrt{2}) = \boxed{14\sqrt{2}}$$

Q8]... **[10 points]** Find the intervals where $f(x)$ is increasing, decreasing, concave up, concave down. Find local maxima, minima and points of inflection. You **don't** have to sketch the graph of f.

$$f(x) = \frac{x}{x^2+3}$$

$$f'(x) = \frac{1(x^2+3) - x(2x)}{(x^2+3)^2}$$

$$= \frac{3-x^2}{(x^2+3)^2}$$

$f'(x) = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

$3-x^2$	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\downarrow$	$\uparrow$	$\downarrow$

$-\sqrt{3}$: local Min

$\sqrt{3}$: local Max

$$f''(x) = \frac{(-2x)(x^2+3)^2 - (3-x^2)\,2(x^2+3)(2x)}{(x^2+3)^4}$$

$$= \frac{-2x^3 - 6x - 12x + 4x^3}{(x^2+3)^3}$$

$$= \frac{2x^3 - 18x}{(x^2+3)^3} = \frac{2x(x^2-9)}{(x^2+3)^3}$$

$f''(x) = 0$

$x = 0, \quad x^2 = 9$

$x = \pm 3$

	$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
f'' $2x(x^2-9)$	$-$	$+$	$-$	$+$
f	CCD	CCU	CCD	CCU

$-3, 0, 3$ are all inflection pts.

Q9]... [10 points] Newton's law of motion states that

$$F = m\frac{dv}{dt}$$

where F is a force acting on a particle of mass m, and $\frac{dv}{dt}$ is the acceleration of the particle.

Use Newton's law of motion to derive a law of conservation of energy for a particle of mass m moving from point A to point B along the x-axis under a spring force $F = -kx$.

$$-kx = m\frac{dv}{dt}$$

$$= m\frac{dv}{dx}\frac{dx}{dt} \quad \text{--- Ch. Rule}$$

$$= mv\frac{dv}{dx} \quad \text{---- velocity, } v = \frac{dx}{dt}.$$

Now take $\int_A^B dx$ of both sides ....

$$\int_A^B -kx\,dx = \int_A^B mv\frac{dv}{dx}\,dx$$

$$= -\frac{kx^2}{2}\Big]_A^B \quad \text{--- power rule}$$

$$= -\tfrac{1}{2}kB^2 + \tfrac{1}{2}kA^2$$

$$= \int_{v_A}^{v_B} mv\,dv \quad \text{--- } v \text{ subst}^{n}.$$

$$= \frac{mv^2}{2}\Big]_{v_A}^{v_B} \quad \text{-- Power Rule}$$

$$= \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2$$

$$\tfrac{1}{2}kA^2 - \tfrac{1}{2}kB^2 = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2$$

$$\Rightarrow \boxed{\tfrac{1}{2}kA^2 + \tfrac{1}{2}mv_A^2 = \tfrac{1}{2}kB^2 + \tfrac{1}{2}mv_B^2}$$

Conservation of energy