SOLUTIONS — UNIV 1000-010 — FA 2007 — MID I

Q1]... [15 points] For each of the following, say if the statement is true or false.

(a) If $f(x)$ and $g(x)$ each have second derivatives, then

$\boxed{\text{FALSE}}$

$$\frac{d^2(fg)}{dx^2} = \frac{d^2 f}{dx^2} g + f\frac{d^2 g}{dx^2}$$

$$(fg)'' = ((fg)')' = (f'g + fg')' \text{ --- product Rule}$$

$$= (f'g)' + (fg')' \text{ --- sum Rule}$$

$$= f''g + f'g' + f'g' + fg'' \text{ --- Product Rule } (\times 2)$$

But this is NOT same as RIGHT HAND SIDE ABOVE!

(b) If the position of a particle at time t is given by $x(t) = t^3 - 3t^2$, then the particle is decelerating (slowing down) during the interval from time $t = 0$ until time $t = 1$.

$\boxed{\text{TRUE}}$

velocity $= x'(t) = 3t^2 - 6t$; acceleration $= x''(t) = 6t - 6$

At $t=0$, $x''(0) = -6$
At $t=1$, $x''(1) = 6-6 = 0$

$\Rightarrow x''(t)$ is $\ominus$ for t between 0 and 1

$\Rightarrow$ decelerating!

(c) If $f(x)$ is differentiable at the point a then

$\boxed{\text{FALSE}}$

$$\lim_{h\to 0} \frac{f(a) - f(a-h)}{h} = -f'(a)$$

$$\lim_{h\to 0} \frac{f(a) - f(a-h)}{h} = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{k\to 0}\left(\frac{f(a+k) - f(a)}{k}\right) = f'(a)$$

$$\neq -f'(a).$$

(d) The piecewise defined function y is continuous at 0

$\boxed{\text{TRUE}}$

$$y = \begin{cases} x\sin(1/x) & \text{when } x < 0 \\ 0 & \text{when } x = 0 \\ x^2\cos(1/x) & \text{when } x > 0 \end{cases}$$

By Squeeze Thm $\lim_{x\to 0^-}(y) = 0$ and $\lim_{x\to 0^+}(y) = 0$

Both of these are equal & are equal to $y(0)$.

(e)

$\boxed{\text{TRUE}}$

$$\lim_{x\to 3} \frac{x^{10} - 3^{10}}{x - 3} = 10(3^9)$$

$$\lim_{x\to 3} \frac{x^{10} - 3^{10}}{x-3} = \left.\frac{d(x^{10})}{dx}\right|_{x=3} = \left.10x^9\right|_{x=3} = 10(3)^9$$

Def$^{\underline{n}}$ of $\frac{dy}{dx}$

Power Rule

Q2]... **[15 points]** Write down the values of the following two limits (you do **not** have to give proofs).

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \leftarrow 1$$

$$\lim_{x \to 0} \frac{1-\cos(x)}{x} = 0 \quad \leftarrow 2$$

Write out the angle addition formula for the cosine function.

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \quad \leftarrow 3$$

Compute the derivative of $\cos(x)$ at the point a using the limit of the difference quotient definition of derivative. Show all your work.

$$\frac{d(\cos x)}{dx}\Big|_{x=a} \underset{\substack{\text{Limit def}^n \\ \text{of } \frac{dy}{dx}}}{=} \lim_{h \to 0} \left(\frac{\cos(a+h) - \cos(a)}{h} \right)$$

by 3 $$= \lim_{h \to 0} \left(\frac{\cos(a)\cos(h) - \sin(a)\sin(h) - \cos(a)}{h} \right)$$

$$= \lim_{h \to 0} \left[\cos(a)\left(\frac{\cos(h)-1}{h}\right) - \sin(a)\left(\frac{\sin(h)}{h}\right) \right]$$

Limit Laws $$= \cos(a) \lim_{h \to 0} \left(\frac{\cos(h)-1}{h} \right) - \sin(a) \lim_{h \to 0} \left(\frac{\sin(h)}{h} \right)$$

by 1 and 2 $$= \cos(a) \cdot 0 - \sin(a) \cdot 1 = \boxed{-\sin a}$$

Q3]... [8 points] Verify that the graphs of $y = x^2$ and $y = \frac{1}{\sqrt{x}}$ intersect at the point $(1, 1)$.

$y = x^2$.
When $x = 1$
$y = (1)^2 = 1$

Graph contains $(1,1)$

Both graphs contain $(1,1)$
So they intersect at $(1,1)$

$y = \frac{1}{\sqrt{x}}$.
When $x = 1$
$y = \frac{1}{\sqrt{1}} = \frac{1}{1} = 1$

Graph contains $(1,1)$.

Show that these graphs are perpendicular at the intersection point $(1, 1)$; that is, show that their tangent lines at the point $(1, 1)$ are perpendicular.

$y = x^2$

Tangent line slope

$$= \left.\frac{dy}{dx}\right|_{x=1}$$

$$= 2x\Big|_{x=1}$$

$$= 2(1) = 2$$

$y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

Tangent Line slope

$$= \left.\frac{dy}{dx}\right|_{x=1}$$

$$= -\frac{1}{2}x^{-3/2}\Big|_{x=1}$$

$$= -\frac{1}{2(\sqrt{1})(1)}$$

$$= -\frac{1}{2}$$

Product of slopes $= (2)(-\frac{1}{2}) = -1$

$\Rightarrow$ Lines are $\perp$

$\Rightarrow$ curves intersect perpendicularly.

Q4]...[12 points] Compute the derivatives $y'$of the following functions. Write down the names of the differentiation rules that you used in each case.

QUOTIENT RULE:

$$y = \frac{(\sin(x) + 4x + 3)}{(x^8 - 5x)}$$

$$y' = \frac{(\cos(x) + 4)(x^8 - 5x) - (\sin(x) + 4x + 3)(8x^7 - 5)}{(x^8 - 5x)^2}$$

(Also sum & power rules).

PRODUCT RULE:

$$y = (\sqrt{x} + x + 7)(x^8 - 5x + 3)$$

$$y' = \left(\frac{1}{2\sqrt{x}} + 1\right)(x^8 - 5x + 3) + (\sqrt{x} + x + 7)(8x^7 - 5)$$

(Also sum + power rules)