Q1]... **[15 points]** Consider the plane P_1 given by the equation $x - 3y + z = 1$.
Verify that the two points $(4, 1, 0)$ and $(2, 0, -1)$ both lie in the plane P_1.

$$(4) - 3(1) + (0) = 4 - 3 = 1 \checkmark$$

$$(2) - 3(0) + (-1) = 2 - 1 = 1 \checkmark$$

Find the equation of the plane P_2 which contains the two points above, and is perpendicular to the plane P_1.

P_2 is $\perp$ $x - 3y + z = 1$ $\Rightarrow$ $\langle 1, -3, 1\rangle$ is parallel to P_2

P_2 contains $(4,1,0)$ & $(2,0,-1)$ $\Rightarrow$ $\langle 4-2, 1-0, 0-(-1)\rangle$ is parallel to P_2

$$\langle 4-2, 1-0, 0-(-1)\rangle = \langle 2, 1, 1\rangle$$

$$\Rightarrow \vec{N} = \langle 1, -3, 1\rangle \times \langle 2, 1, 1\rangle$$ is a Normal for P_2

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \langle -4, 1, 7\rangle$$

$$\boxed{-4(x-4) + 1(y-1) + 7(z-0) = 0}$$ Eqn of P_2

Q2]... **[15 points]** Find the equation of the tangent **line** to the vector curve

$$\mathbf{r}(t) = \langle 2\cos(t), 3\sin(t), -4t\rangle$$

at the point where $t = \pi/2$.

$$\vec{r}'(t) = \langle -2\sin t, 3\cos t, -4\rangle$$

$$\vec{r}'(\pi/2) = \langle -2(1), 3(0), -4\rangle$$

$$= \langle -2, 0, -4\rangle$$

$$\vec{r}(\pi/2) = \langle 0, 3, -4\pi/2\rangle = \langle 0, 3, -2\pi\rangle$$

$$\boxed{\vec{r} = \langle 0, 3, -2\pi\rangle + t\langle -2, 0, -4\rangle}$$

$$\boxed{\begin{aligned} x &= -2t \\ y &= 3 \\ z &= -2\pi - 4t \end{aligned}}$$

Q3]... **[15 points]** Find the equation of a plane which satisfies both of the following conditions: (1) it contains the line of intersection of the two planes $2x - 2y + z = 1$ and $3x + 4z = 8$, and (2) it bisects the angle between these two planes.

There are two possible planes satisfying both conditions (1) and (2) above. I am happy with either one.

Step 1. Find a point on the plane:

$\boxed{\text{set } x = 0}$

$$-2y + z = 1$$
$$4z = 8 \leftarrow z = 2$$

$(0, \frac{1}{2}, 2)$

$$-2y = -1 \qquad y = \frac{1}{2}$$

step 2. Find a Normal vector to the plane:

Normal bisects the normal $\langle 2, -2, 1\rangle$ to $2x-2y+z=1$

& the normal $\langle 3, 0, 4\rangle$ to $3x+4z=8$.

Length of $\langle 2, -2, 1\rangle = \sqrt{2^2+(-2)^2+1^2} = 3$

Length of $\langle 3, 0, 4\rangle = \sqrt{3^2+0^2+4^2} = 5$

$\Rightarrow$ Bisector of $\langle 2,-2,1\rangle$ & $\langle 3,0,4\rangle$ is

$$5\langle 2,-2,1\rangle + 3\langle 3,0,4\rangle$$
$$= \langle 19, -10, 17\rangle$$

We did a Question like this on Hwk!

$|\vec{u}|\vec{v} + |\vec{v}|\vec{u}$ bisects $\vec{u}, \vec{v}$

Step 3. Plane

$$\boxed{19(x-0) - 10\left(y-\tfrac{1}{2}\right) + 17(z-2) = 0}$$

Q4]... **[15 points]** If the vector $\mathbf{r}(t)$ is differentiable and has constant length C, then show that $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$.

$$\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = C^2$$

$$\frac{d}{dt} \Rightarrow \quad 2\,\vec{r}(t)\cdot\frac{d\vec{r}}{dt}(t) = \frac{d}{dt}(C^2) = 0$$

$$\Rightarrow \quad \vec{r}(t)\cdot\frac{d\vec{r}(t)}{dt} = 0$$

$$\Rightarrow \quad \frac{d\vec{r}(t)}{dt} \perp \vec{r}(t).$$

If the acceleration $\mathbf{r}''(t)$ of a particle in 3–dimensions is always parallel to $\mathbf{r}(t)$, then show that the particle is confined to move in a plane.

$$\frac{d}{dt}\left(\vec{r}(t)\times\vec{r}'(t)\right) = \vec{r}'(t)\times\vec{r}'(t) + \vec{r}(t)\times\vec{r}''(t)$$

$$= \vec{0} + \vec{0} \quad \swarrow \text{ Told } \vec{r}''(t) \parallel \vec{r}(t)$$

$$= \vec{0}$$

$$\Rightarrow \quad \vec{r}(t)\times\vec{r}'(t) = \vec{H} \text{ ... a constant vector.}$$

$\Rightarrow$ $\vec{r}(t)$ lies in plane (through $\vec{0}$) with Normal vector $\vec{H}$. $\Rightarrow$ particle moves in this plane.

Q5]... [15 points] Consider the ellipse in the xy–plane, described as the vector curve

$$\mathbf{r}(t) \;=\; a\cos(t)\mathbf{i} + b\sin(t)\mathbf{j}$$

where $a > b > 0$.

Find an expression for the curvature $\kappa(t)$ of the ellipse at the point $\mathbf{r}(t)$.

We're Told

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{|\langle 0, 0, ab\rangle|}{|\langle -a\sin t, b\cos t, 0\rangle|^3}$$

$$= \frac{ab}{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}}$$

$\vec{r}(t) = \langle a\cos t, b\sin t, 0\rangle$

$r'(t) = \langle -a\sin t, b\cos t, 0\rangle$

$r''(t) = \langle -a\cos t, -b\sin t, 0\rangle$

$r' \times r'' = \langle 0, 0, ab(1)\rangle$

$\sin^2 t + \cos^2 t = 1$

Find the maximum and minimum values of $\kappa(t)$ and find the points on the ellipse where these occur. You will need to draw a sketch of the ellipse for this.

$$K = \frac{ab}{[a^2\sin^2 t + b^2(1-\sin^2 t)]^{3/2}} = \frac{ab}{[(a^2-b^2)\sin^2 t + b^2]^{3/2}}$$

Clearly $\begin{cases} \text{Max } K \text{ when } \sin^2 t = 0 & (\text{denom} = \text{min}), \quad t = 0, \pi. \\ \text{Min } K \text{ when } \sin^2 t = 1 & (\text{denom} = \text{Max}), \quad t = \pi/2, \tfrac{3\pi}{2}. \end{cases}$

$$\boxed{K_{max} = \frac{a}{b^2}} \qquad \boxed{K_{min} = \frac{b}{a^2}}$$