Math 2433–006 Honors Calculus III

Extra Hwk III

Deriving Newton's Law of Gravitation from Kepler's Laws

Setup: We're given a planet acted upon by gravity (of a sun). All we assume about the gravitational force is that it depends on the position of the planet. So, taking the sun as the origin of our coordinate system, we can write $\mathbf{F}(\mathbf{r})$ for the force of gravity of the sun acting on the planet, where $\mathbf{r}$ denotes the position vector of the planet.

We want to end up showing that:

- $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\hat{\mathbf{r}}$. That is $\mathbf{F}$ is a central force (and so acts parallel to $\hat{\mathbf{r}}$).
- $f(\mathbf{r}) = -\frac{K}{r^2}$ where K is some positive constant, and where r is the magnitude of $\mathbf{r}$. Thus, the minus sign indicates an attracting force, and the $\frac{1}{r^2}$ gives the inverse square law which is key to Newton's Law of Gravitation.

We'll only need to use Newton's Law of Motion (*force equals mass times acceleration*) together with the first two of Kepler's Laws (which were formulated as the result of lots of observational data):

I... Planets move in elliptical orbits (which are by definition planar!) about the sun, with the sun at a focus.

II... Planets sweep out equal areas in equal times.

We begin by reminding ourselves of the setup. We'll use cylindrical coordinates (r, θ, z) with the sun at the origin, and the planet moving in an ellipse (with one focus at the origin) in the (r, θ)-plane. So the first thing to do is to write down the mathematical versions of Kepler's Laws I and II. Here they are:

MATH-I... $r = \frac{ed}{1+e\cos\theta}$ (where, as in section 11.7, e denotes the eccentricity of ellipse, and d denotes the distance from the sun to the directrix of the ellipse)

MATH-II... $\frac{d}{dt}\int \frac{r^2\,d\theta}{2} = C$ (a constant), or in other words

$$r^2\dot{\theta} = 2C$$

Note that we're using the (physics) "dot" shorthand to denote differentiation with respect to time t. So $\dot{r} = \frac{dr}{dt}$, $\ddot{r} = \frac{d^2r}{dt^2}$, $\dot{\theta} = \frac{d\theta}{dt}$ and $\ddot{\theta} = \frac{d^2\theta}{dt^2}$.

Before we begin, it's good (for our souls) to recall the unit vectors

$$\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

and

$$\hat{\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

Now we're ready for the problem.

1. Prove that

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\theta}$$

   and that

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{\mathbf{r}}$$

2. Prove that the velocity vector becomes

$$\dot{\mathbf{r}} = \frac{d(r\hat{\mathbf{r}})}{dt} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$$

   when expressed in terms of $\hat{\mathbf{r}}$ and $\hat{\theta}$.

3. Prove that the acceleration vector becomes

$$\ddot{\mathbf{r}} = (\ddot{r} - r(\dot{\theta})^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

   when expressed in terms of $\hat{\mathbf{r}}$ and $\hat{\theta}$.

4. Use $\mathbf{r} = r\hat{\mathbf{r}}$, the expression in 2 above, and Kepler's second law (**MATH-II**) to obtain

$$\mathbf{r} \times \dot{\mathbf{r}} = 2C\mathbf{k}$$

5. Differentiate 4 above and deduce that $\ddot{\mathbf{r}}$ is parallel to $\mathbf{r}$. Thus, by Newton's law of motion, $\mathbf{F}(\mathbf{r})$ which is mass times $\ddot{\mathbf{r}}$ is also parallel to $\mathbf{r}$. Thus, $\mathbf{F}$ is a central force.

6. By 5 above we can write $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\hat{\mathbf{r}}$. We know from 3 and Newton's law of motion, that we can also write

$$\mathbf{F}(\mathbf{r}) = m\ddot{\mathbf{r}} = m(\ddot{r} - r(\dot{\theta})^2)\hat{\mathbf{r}} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

   Comparing these two expressions for $\mathbf{F}$ we see that the $\hat{\theta}$ expression vanishes, and that

$$\mathbf{F} = m\ddot{\mathbf{r}} = m(\ddot{r} - r(\dot{\theta})^2)\hat{\mathbf{r}}$$

7. Finally, use Kepler's first law (**MATH-I**) to write down r, and hence to compute $\dot{r}$ and $\ddot{r}$. It'll help to use **MATH-II** to substitute in for $\dot{\theta}$ after each successive differentiation. You should (if I have not made any calculation errors!) end up with something like

$$f(\hat{\mathbf{r}}) \;=\; \frac{-\frac{4C^2}{ed}}{r^2}$$

which is indeed of the form $\frac{-K}{r^2}$ where K is some positive constant. Thus, we have deduced an attracting inverse square law central force from Kepler I and II. Go and enjoy a Fig Newton...