Miscellaneous Vector Applications

1. Rotation of co-ordinate axes, and the general form of the conic in Cartesian coordinates.
In this section, you will use vectors to understand how Cartesian co-ordinates change when one rotates the co-ordinate axes through an angle θ. This information will then be used to see how the locus of points in the Cartesian plane which satisfy a general quadratic

$$Ax^2 + By^2 + Cxy + Dx + Ey + F \ = \ 0$$

with at least one of A, B, C non-zero is a conic.

Q1. Let $\widehat{\mathbf{u}}$ be the unit vector which is obtained from the standard basis vector $\mathbf{i} = \langle 1, 0\rangle$ by a rotation through θ about the origin. Similarly, let $\widehat{\mathbf{v}}$ be the unit vector which is obtained from the standard basis vector $\mathbf{j} = \langle 0, 1\rangle$ by a rotation through θ about the origin. Find the coordinates of the vectors $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$.

Q2. We can use $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ as two basis vectors, and write an arbitrary point P (position vector $\mathbf{P}$) in the plane as being a sum $u\widehat{\mathbf{u}} + v\widehat{\mathbf{v}}$ for suitable scalars u, v. These scalars are called the *co-ordinates* of $\mathbf{P}$ with respect to the basis vectors $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$. We can also write $\mathbf{P}$ as a combination $x\mathbf{i} + y\mathbf{j}$. Find expressions for x and y in terms of u, v. These expressions are capture the change of coordinates when one passes from u–axis v–axis description to the standard x–axis y–axis description.

Q3. Now consider the expression

$$Ax^2 + By^2 + Cxy + Dx + Ey + F \ = \ 0$$

We know that not all of A, B, C are zero (otherwise this is a linear equation).
If $C = 0$ show that you can convert this into the standard form of a conic (by completing squares). You may have to make some assumptions on F.
What conic do you get if $A = B$? What conic do you get if one of A or B is zero?

Q4. So we now need to consider the equation above where $C \neq 0$. Use your expressions in **Q2** to write out what this equation becomes in the uv–plane. Show that it is possible to select θ carefully so that the equation transforms to

$$A_1u^2 + B_1v^2 + D_1u + E_1v + F_1 \ = \ 0$$

You should express θ as a function of A, B, C.
Now, you are back in the case of **Q3**. The only difference is that you are looking a the locus of points from the perspective of the u– and v–axes.

Q5. Run the analysis above for the graph of the reciprocal function, $y = 1/x$. Rewrite the function as $xy = 1$, draw the graph. Use your answer in **Q4** to find a value of θ. Draw in the u–axis and the v–axis on your diagram. Rewrite the equation in uv–co-ordinates. Do you recognize this equation?

Summary. You now know 4 distinct descriptions of a conic.

(1) As a section of a cone in 3–dimensions by a plane.
(2) As a locus of points whose distance from a fixed point (focus) is a constant multiple (eccentricity) of the distance to a fixed line (directrix).
(3) As a locus of points whose distances from a pair of fixed points (foci) satisfy some algebraic identity. Sum = constant (ellipse). Sum = constant and foci coincide (circle). Difference = constant (hyperbola).
(4) As a locus of points satisfying a general quadratic equation $Ax^2 + By^2 + Cxy + Dx + Ey + F \ = \ 0$.

2. Permanently happy and occasionally co-planar ants.

You happen to own an "ant farm", and decide to create a playground for your four favorite ants: (A)lice, (B)ob, (C)huck and (D)aphne. You start with 4 towers, and tie four pieces of copper wire between the tops of the towers as shown. The towers are of various heights, so the tops P_1, P_2, P_3 and P_4 are four non-coplanar points.

At time $t = 0$ you place A at point P_1, B at point P_2, C at point P_3, and D at point P_4. Each ant starts walking/sliding/running off along the copper wire which connects his/her tower to the next tower in the sequence; 1 to 2 to 3 to 4 back to 1. At time $t = 1$ (one minute later), (A)lice is at P_2, (B)ob is at P_3, (C)huck is at P_4 and (D)aphne is at P_1. Prove that at some time t during this minute all four ants are coplanar.

Hint. We talked a bit about this in class. Let $A(t) = \langle a_1(t), a_2(t), a_3(t)\rangle$ be the position of (A)lice at time t, and similarly for $B(t)$, $C(t)$ and $D(t)$. You can assume that all twelve coefficient functions are continuous functions of t. You have been told that $A(0) = P_1$, $A(1) = P_2$ etc.

You have a way of testing co-planarity of four points using vectors.