MATH 2513 - 001, FA '09.

Worked problems from the Cardinality Handout.

Examples 18. (i) We've seen several explicit bijections $\mathbb{Z}^+ \times \mathbb{Z}^+ \xrightarrow{b} \mathbb{Z}^+$.

(ii) Let $f: \mathbb{Z} \to \mathbb{Z}^+$ be a bijection

E.g. $f(x) = \begin{cases} 2|x| & \text{if } x < 0 \\ 2x+1 & \text{if } x \geq 0 \end{cases}$

Then the composition $b \circ (f \times f): \mathbb{Z} \times \mathbb{Z} \xrightarrow[f \times f]{} \mathbb{Z}^+ \times \mathbb{Z}^+ \xrightarrow[b]{} \mathbb{Z}^+$ is a bijection.

(iii) First show $\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$ (n copies) is countably infinite.

$(n=3)$ $(\mathbb{Z}^+ \times \mathbb{Z}^+) \times \mathbb{Z}^+ \xrightarrow[b \times \mathbb{1}_{\mathbb{Z}^+}]{} \mathbb{Z}^+ \times \mathbb{Z}^+ \xrightarrow[b]{} \mathbb{Z}^+$

$b \circ (b \times \mathbb{1}_{\mathbb{Z}^+})$ works!

(inductive step) Suppose $b_k : \overbrace{\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+}^{k\text{-copies}} \longrightarrow \mathbb{Z}^+$ is a bijection

then

$$\underbrace{(\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+) \times \mathbb{Z}^+}_{k+1 \text{ copies}} \xrightarrow{b_k \times \mathbb{1}_{\mathbb{Z}^+}} \mathbb{Z}^+ \times \mathbb{Z}^+ \xrightarrow{b} \mathbb{Z}^+$$

the composite $b \circ (b_k \times \mathbb{1}_{\mathbb{Z}^+})$ works.

Now show $\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ copies}}$ is countable

$$\mathbb{Z} \times \cdots \times \mathbb{Z} \xrightarrow[f \times \cdots \times f]{} \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+ \xrightarrow[b_n]{} \mathbb{Z}^+$$

composite works!

(iv) We'll find an injection $\mathbb{Q}^+ \longrightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$ and then use Theorem 20 to get that $\mathbb{Q}^+$ is countable. Since $\mathbb{Z}^+ \subset \mathbb{Q}^+$, we conclude $\mathbb{Q}^+$ is infinite $\Rightarrow$ countably infinite.

$$f : \mathbb{Q}^+ \longrightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$$
$$: r \longmapsto (p, q)$$

where $r = \frac{p}{q}$ is rational written in lowest terms as a ratio of 2 integers (p, q have no factors in common).

f is injective

$$f(r_1) = f(r_2) \Rightarrow (p_1, q_1) = (p_2, q_2)$$
$$\Rightarrow p_1 = p_2 \;\&\; q_1 = q_2$$
$$\Rightarrow \frac{p_1}{q_1} = \frac{p_2}{q_2} \Rightarrow r_1 = r_2$$

(v) $\mathbb{Q}^- \longrightarrow \mathbb{Q}^+$ is a bijection
$x \longmapsto -x$

$\Rightarrow$ $\mathbb{Q}^-$ & $\mathbb{Q}^+$ are both countable, so is $\{0\}$

$\Rightarrow$ $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$ is countable by 19(d).

$\mathbb{Q} \supseteq \mathbb{Z}^+ \Rightarrow \mathbb{Q}$ infinite

$\Rightarrow$ $\mathbb{Q}$ countably infinite.

Examples 2.1:

$$\left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \xrightarrow{\alpha} \mathbb{R}$$
$$x \longmapsto \tan(x)$$
is a bijection.

use calc/trig to check this

eg: injective since
$\frac{d}{dx}\tan(x) = \sec^2(x) > 0$
$\Rightarrow$ $\tan(x)$ strictly increasing
$\Rightarrow$ $\tan(x)$ injective.

$$\left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \xrightarrow{\beta} (0,\pi)$$
$$x \longmapsto x + \tfrac{\pi}{2}$$
is bijection ($x \mapsto x - \pi/2$ is inverse!)

$$(0,\pi) \xrightarrow{\gamma} (0,1)$$
$$x \longmapsto \frac{x}{\pi}$$
is bijection ($x \mapsto \pi x$ is inverse!)

$$(0,1) \xrightarrow{\delta} (a,b) \qquad \boxed{a<b}$$
$$x \longmapsto (b-a)x + a$$
is bijection ($x \mapsto \frac{x-a}{b-a}$ is inverse)

$$\mathbb{R} \xrightarrow{\epsilon} (0,\infty)$$
$$x \longmapsto e^x$$
is bijection ($x \mapsto \ln(x)$ is inverse).

Note that the map δ extends to a bijection

$$[0,1) \to [a,b), \quad (0,1] \to (a,b], \quad [0,1] \to [a,b]$$

so we only have to show $(0,1] \longleftrightarrow [0,1) \longleftrightarrow [0,1]$ are all bijective to $(0,1)$.

Hilbert Hotel trick works here ... we just need to find a copy of H-Hotel inside $(0,1)$

eg $A = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\} \subseteq \{0,1\}$

Define $f : (0,1] \to (0,1)$

$$x \longmapsto \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \in A \\ x & \text{if } x \notin A \cup \{1\} \end{cases}$$

check that it's a bijection

Define $g : [0,1] \longrightarrow (0,1)$

$$x \longmapsto \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x = 1 \\ \frac{1}{n+2} & \text{if } x = \frac{1}{n} \in A \\ x & \text{if } x \notin A \cup \{0,1\} \end{cases}$$

check this is a bijection

check that $[0,1) \to (0,1] : x \mapsto 1 - x$ is a bijection

__Examples 24__ By 18, $\mathbb{Q}$ is countable.

By 22, $\mathbb{R}$ is uncountable.

Let $\mathbb{I}$ = set of irrational numbers.

Note $\mathbb{I} \cup \mathbb{Q} = \mathbb{R}$ (by defn of irrational!)

If $\mathbb{I}$ were countable then 19(d) $\Rightarrow$ $\mathbb{R}$ countable

$\Rightarrow$ contradiction

$\Rightarrow$ $\mathbb{I}$ must be uncountable.

__Examples 26.__

$$X = \sqrt{2}$$

$$X^2 = 2$$

$$X^2 - 2 = 0$$

$X = \sqrt{2}$ is a root of the polynomial $x^2 - 2$

$$\underbrace{1}_{\in \mathbb{Z}} . x^2 + \underbrace{0}_{\in \mathbb{Z}} . x + \underbrace{(-2)}_{\in \mathbb{Z}}$$

$\Rightarrow$ $\sqrt{2}$ is __algebraic__.

Given $x = \frac{p}{q} \in \mathbb{Q}$ $\Rightarrow$ $qx = p$

$$\Rightarrow \underbrace{q}_{\in \mathbb{Z}} x + \underbrace{(-p)}_{\in \mathbb{Z}} = 0$$

$\Rightarrow$ x is root of this poly

$\Rightarrow$ x algebraic.

Let $A = \{x \in \mathbb{R} \mid x \text{ is algebraic}\}$. To show A is countable.

$x \in A \Rightarrow x$ is a root of some polynomial $a_0 + a_1 x + \cdots + a_n x^n$ where $a_i \in \mathbb{Z}$.

$$P_n = \{\text{polynomials of degree} \leq n \text{ with integer coefficients}\}$$
$$= \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{Z}\}$$

Note
$$P_n \longrightarrow \mathbb{Z}^{n+1}$$
$$a_0 + \cdots + a_n x^n \longmapsto (a_0, a_1, \_\_\_, a_n)$$
is a bijection.

<u>Algebra fact</u> (*) $\boxed{\text{poly of degree} \leq n \text{ has } \leq n \text{ real roots.}}$

Let $A_n = \{x \in \mathbb{R} \mid x \text{ is a root of a poly in } P_n\}$

Note A_n is countable, since each poly has at most n real roots & there are countably many polys

$\Rightarrow A_n = $ countable union of $\underset{\substack{\uparrow \\ \text{at most } n \text{ elements}}}{\underline{\text{finite}}}$ sets of numbers

$\Rightarrow A_n$ countable $\leftrightarrow \boxed{19(d)}$

Finally, $A = \bigcup_{n \geq 1}^{\infty} A_n =$ countable union of countable sets $\Rightarrow$ countable $\longleftrightarrow \boxed{19(d)}$.

Therefore $\mathbb{R} - \mathbb{A}$ = set of transcendental #'s is uncountable!

know $\mathbb{R} = \mathbb{A} \cup (\mathbb{R} - \mathbb{A})$

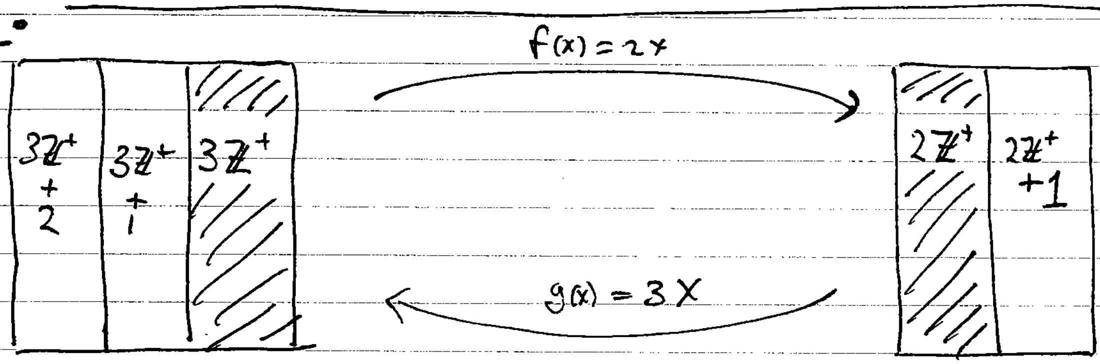
$\mathbb{R}$: uncountable

$\mathbb{A}$: countable

$\mathbb{R} - \mathbb{A}$: If this were countable, then 19(b) $\Rightarrow$ $\mathbb{R}$ countable $\Rightarrow$ contradiction!

EXAMPLE 31.

$A = \mathbb{Z}^+$



$\mathbb{Z}^+ = B$

Family trees with $b \in B - f(A)$ as "original ancestor"

$\mathbb{Z}^+ - 2\mathbb{Z}^+ \Rightarrow$ odd #s ...

are as follows ...

$$\begin{array}{cccc}
1 & 3 & 5 & 7 \\
\downarrow g & \downarrow g & \downarrow g & \downarrow g \\
3 & 9 & 15 & 21 \\
\downarrow f & \downarrow f & \downarrow f & \downarrow f \\
6 & 18 & 30 & 42 \\
\downarrow g & \downarrow g & \downarrow g & \downarrow g \\
18 & 54 & 90 & 126 \\
\downarrow f & \downarrow f & \downarrow f & \downarrow f \\
36 & 108 & 180 & 252 \\
\downarrow g & \downarrow g & \downarrow g & \downarrow g
\end{array}$$

Elements of A are of the form

$$2^a 3^{b+1} y$$

where $\begin{cases} \bullet\ a \geq 0, \quad b \geq a \\ \bullet\ 2 \nmid y \\ \bullet\ 3 \nmid y \end{cases}$

$$A \longrightarrow B$$

$$h : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$$

$$: x \longmapsto \begin{cases} \dfrac{x}{3} & \text{if } x = 2^a 3^{b+1} m \text{ where } a \geq 0,\ b \geq a \text{ \& } m \text{ not divisible by either 2 or 3.} \\ 2x & \text{otherwise.} \end{cases}$$

It's a crazy bijection ↗
(but a bijection nonetheless!)

Exercise 34. Show $\mathbb{R}^2$ and $\mathbb{R}$ have same cardinality.

By Examples 21, we know $\exists$ bijection $b : \mathbb{R} \to (0,1)$

then $b \times b : \mathbb{R}^2 \longrightarrow (0,1)^2$ is also a bijection.

So ~~the~~ problem is equivalent to asking for a bijection

$$(0,1)^2 \longrightarrow (0,1).$$

We'll show that such a bijection exists by first exhibiting

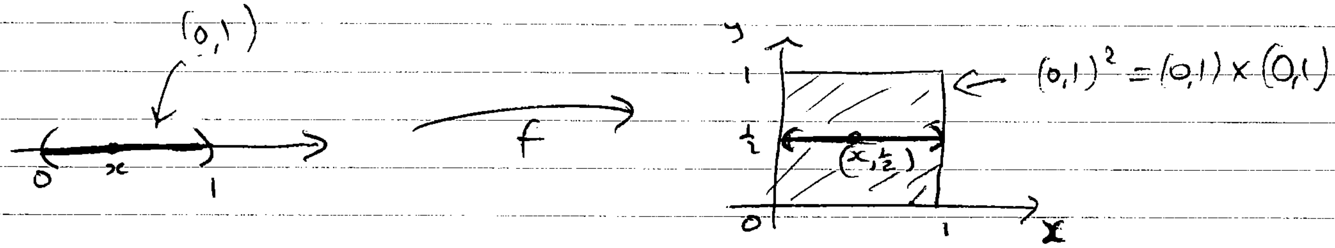
- an injection $(0,1) \longrightarrow (0,1)^2$
- an injection $(0,1)^2 \longrightarrow (0,1)$

& then using Schröder – Bernstein theorem (#30).

An injective map $(0,1) \xrightarrow{f} (0,1)^2$

$$x \longmapsto (x, \tfrac{1}{2})$$

$$f(x_1) = f(x_2) \Rightarrow (x_1, \tfrac{1}{2}) = (x_2, \tfrac{1}{2})$$
$$\Rightarrow x_1 = x_2 \qquad \Rightarrow f \text{ injective!}$$



An injective map $(0,1)^2 \longrightarrow (0,1)$

Number theory fact : Each $x \in (0,1)$ admits a <u>unique</u> decimal expansion which does <u>not</u> end in an infinite string of 9's.

use this for free!

$$x \in (0,1) \rightsquigarrow 0.a_1 a_2 a_3 \ldots\ldots$$

a_i uniquely determined

a_n not eventually constant string of 9's.

$$g : (0,1)^2 \longrightarrow (0,1)$$
$$: (0.a_1 a_2 \ldots,\ 0.b_1 b_2 \ldots\ldots) \longmapsto 0.a_1 b_1 a_2 b_2 \ldots\ldots$$

<u>Check</u> g is well-defined, injective.

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g is well-defined since the expansions $0.a_1a_2\ldots$ and $0.b_1b_2\ldots$ are UNIQUELY determined by the pair of numbers $(x,y)\in(0,1)^2$.

[We're avoiding situations like this $x = 0.5$
$y = 0.333\ldots$

write $x = 0.50000\ldots$ & $y = 0.333\ldots$ $\Rightarrow$ $g(x,y) = 0.530303\ldots$

write $x = 0.49999\ldots$ & $y = 0.333\ldots$ $\Rightarrow$ $g(x,y) = 0.43939393\ldots$

these two #s are distinct!

Rule out this problem by ruling out the $0.4999\ldots$ expansion for $x = \frac{1}{2}$.]

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g is injective. $g(x,y) = g(s,t)$

write $x = 0.a_1a_2\ldots$ $\quad s = 0.c_1c_2\ldots$
$y = 0.b_1b_2\ldots$ $\quad t = 0.d_1d_2\ldots$

$\Rightarrow \quad 0.a_1b_1a_2b_2\ldots = 0.c_1d_1c_2d_2\ldots$

& there's no tail of $99\ldots$'s.

$\Rightarrow \quad a_1 = c_1, \quad b_1 = d_1$
$\quad a_2 = c_2, \quad b_2 = d_2$
$\quad \vdots \qquad \vdots$

$\Rightarrow \quad 0.a_1a_2\ldots = 0.c_1c_2\ldots$ & $0.b_1b_2\ldots = 0.d_1d_2\ldots$
$\Rightarrow \quad x = s$ & $y = t$ $\Rightarrow (x,y) = (s,t)$
$\Rightarrow$ injective.