Solutions to "handout" questions on cardinality.

Q1

$$\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & \cdots \\
(2,1) & (2,2) & (2,3) & \cdots \\
(3,1) & (3,2) & \cdots & \\
\vdots & & &
\end{array}$$

The bijection $b : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ ~~defined~~ indicated by the diagram above was shown in class to have the explicit form:

$$b : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$$
$$: (m, n) \longmapsto \frac{(m+n)(m+n-1)}{2} - m + 1$$

These may be swapped!

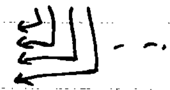
(a) So the bijection $b_2 : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$

given by reversing the arrows in the diagram

has the explicit form

$$(m, n) \longmapsto \frac{(m+n)(m+n-1)}{2} - n + 1$$

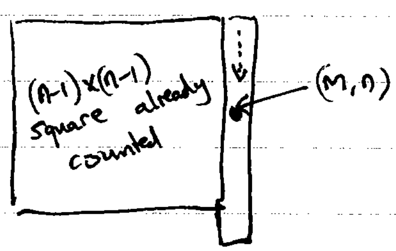
(b) The bijection $b_3 : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ indicated by the diagram



has the following formula.

$$b_3(m,n) = \begin{cases} (n-1)^2 + m & \text{if } n \geq m \\ (m-1)^2 + 2m - n & \text{if } n \leq m \end{cases}$$

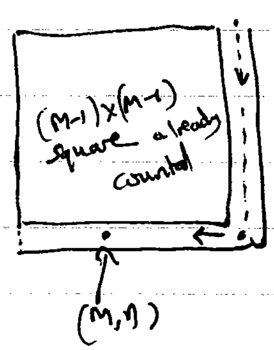
CASE : $m \leq n$



(m,n) is m^{th} element counted after we have counted everything in the previous $(n-1)\times(n-1)$ square.

$\Rightarrow (n-1)^2 + m$

CASE : $m \geq n$



(m,n) is $m + (m-n)$ on the list after we have counted everything on the previous $(m-1)\times(m-1)$ square.

$\Rightarrow (m-1)^2 + 2m - n$

Q2 $\quad A = \{f \mid f:\{0,1\} \to \mathbb{Z}^+\} = {\mathbb{Z}^+}^{\{0,1\}}$

Claim: The map $\varphi: A \longrightarrow (\mathbb{Z}^+) \times (\mathbb{Z}^+)$

$$f \longmapsto (f(0), f(1))$$

is a bijection from A to $\mathbb{Z}^+ \times \mathbb{Z}^+$

$$\Rightarrow |A| = |\mathbb{Z}^+ \times \mathbb{Z}^+| = |\mathbb{Z}^+| = \aleph_0$$

Hilbert Hotel trick.

$\Rightarrow$ A is countable

$$B_n = \{f \mid f:\{1, \_\_, n\} \to \mathbb{Z}^+ \text{ function}\}$$

Claim: The map $\psi_n : B_n \longrightarrow \underbrace{\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+}_{n \text{ factors}}$

$$: f \longmapsto (f(1), \_\_, f(n))$$

is a bijection

$$\Rightarrow |B_n| = |\underbrace{\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+}_{n}| = |\mathbb{Z}^+| = \aleph_0$$

$\Rightarrow$ B_n countable.

$$C = \bigcup_{n=1}^{\infty} B_n = \text{countable union of countable sets}$$

$\Rightarrow$ C is countable by # 19(d).

$$D = \{ f \mid f : \mathbb{Z}^+ \to \mathbb{Z}^+ \text{ a function} \}$$

$$\supseteq \{ f \mid f : \mathbb{Z}^+ \to \{1, 2\} \text{ a function} \} = D'$$

view element of D' as an element of D
by extending the codomain from $\{1,2\}$
to all of $\mathbb{Z}^+$:

$$\begin{tikzcd} \mathbb{Z}^+ \arrow[r, "f \in D'"] \arrow[rr, bend right, "i \circ f \in D."'] & \{1, 2\} \arrow[r, "i"] & \mathbb{Z}^+ \end{tikzcd}$$

Now $D' = \{1, 2\}^{\mathbb{Z}^+}$

$$\begin{aligned} \psi : \mathbb{P}(\mathbb{Z}^+) &\longrightarrow D' \\ : A &\longmapsto f_A \end{aligned}$$

where
$$\begin{aligned} f_A : \mathbb{Z}^+ &\longrightarrow \{1, 2\} \\ : n &\longmapsto \begin{cases} 1 & \text{if } n \in A \\ 2 & \text{if } n \notin A \end{cases} \end{aligned}$$

ψ is a bijection from $\mathbb{P}(\mathbb{Z}^+)$ to D'.

$\Rightarrow$ D' ~~must~~ is be uncountable.

But $D' \subseteq D$ $\Rightarrow$ D must be uncountable too
(use #20).

$$E = \{ f \mid f : \mathbb{Z}^+ \to \{0,1\} \text{ a function} \}$$

$$= \mathbb{P}(\mathbb{Z}^+)$$

usual bijection

$A \in \mathbb{P}(\mathbb{Z}^+)$

$\Rightarrow A \subseteq \mathbb{Z}^+$

$\longrightarrow \chi_A \in E.$

this map is bijection.

$\Rightarrow E$ is uncountable.

$$F = \{ f \mid f : \mathbb{Z}^+ \to \{0,1\} \text{ such that } f \text{ is eventually } 0 \}$$

Define $F_1 = \{ f \mid f : \mathbb{Z}^+ \to \{0,1\},\ f(i) = 0,\ \forall i > 1 \}$

$F_2 = \{ f \mid f : \mathbb{Z}^+ \to \{0,1\},\ f(i) = 0,\ \forall i > 2 \}$

$\vdots$

$F_m = \{ f \mid f : \mathbb{Z}^+ \to \{0,1\},\ f(i) = 0,\ \forall i > m \}$

Check $|F_1| = 2$ $\quad |F_2| = 4$ $\quad |F_m| = 2^m$.

$F = \bigcup_{m=1}^{\infty} F_m$ is a countable union of (finite) countable sets $\Rightarrow F$ is countable by #19(d).

$$G = \{ f \mid f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+ \text{ an } \underline{\text{injective}} \text{ function} \}.$$

<u>Claim</u> • G is uncountable.

2 things

(i) $|G|$ ~~is~~ not finite (i.e. $\exists$ injection $\mathbb{Z}^+ \to G$).

(ii) $|G|$ not countably infinite (i.e. $\nexists$ bijection $\mathbb{Z}^+ \to G$).

(i)

$$\mathbb{Z}^+ \xrightarrow{\ \Psi\ } G$$
$$n \longmapsto \Psi(n) = f_n$$

$$f_n : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$$
$$: x \longmapsto x+n$$

clearly f_n is injective: $f_n(x) = f_n(y) \Rightarrow x + \cancel{n} = y + \cancel{n}$
$\Rightarrow x = y$.

$\Rightarrow f_n \in G \quad \forall n$.

clearly Ψ is injective: $\Psi(n) = \Psi(m)$
$\Rightarrow f_n = f_m$
$\Rightarrow f_n(1) = f_m(1)$
$\Rightarrow \cancel{1}+n = \cancel{1}+m$
$\Rightarrow n = m$.

$\Rightarrow G$ is an infinite set.

(ii) Suppose $\Phi : \mathbb{Z}^+ \longrightarrow G$ is any function

We'll show ϕ is not surjective $\Rightarrow$ ϕ not bijective

$\Rightarrow \nexists$ bij $\mathbb{Z}^+ \to G$

This & (i) $\Longrightarrow$ G uncountable.

Consider $\phi(\mathbb{Z}^+)$ ....

Outputs

$\phi(1) = f_1 \quad : \quad f_1(1),\ f_1(2),\ f_1(3), \ldots$

$\phi(2) = f_2 \quad : \quad f_2(1),\ f_2(2),\ f_2(3), \ldots$

$\vdots \qquad\qquad \vdots$

Consider the map $g: \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ defined (inductively) as follows

- $g(1) \neq f_1(1)$
- $g(k) \notin \{ g(1), \longrightarrow g(k-1), f_k(k) \}$

Clearly g is injective map $\mathbb{Z}^+ \to \mathbb{Z}^+$ (by def^{n})

$\Rightarrow g \in G$.

Also $g \neq f_n$ for any n (since $g(n) \neq f_n(n)$).

$\Rightarrow g \notin$ Image of $\phi \quad \Rightarrow \phi(\mathbb{Z}^+) \subsetneqq G$.

done! $\square$

Solns to Textbook q=s on cardinality.

Page 163

Q45 $P = \{$ computer programs in C++ $\}$ is countable.

Idea

Let K = alphabet of keystrokes

$= \{A, a, B, b, \ldots, \emptyset, 1, \ldots, 9, \$, \#, \ldots\}$

K is some finite set of ascii characters.

$S_1 = \{$ strings of ascii characters of length 1 $\}$

$S_m = \{$ strings of ascii characters of length m $\}$

A given computer program written in C++ is just a finite file of code $\Rightarrow$ is a finite string of ascii characters (which happens to be compilable by C++ compiler!).

$\Rightarrow$ given program $\in S_m$ for some m.

$$\Rightarrow \quad P \subseteq \bigcup_{m=1}^{\infty} S_m$$

Note $|S_1| = |K|$, $|S_m| = |K|^m$ are all finite, hence countable

$\Rightarrow \bigcup_{m=1}^{\infty} S_m$ is countable union of countable sets $\Rightarrow$ countable by # 19(d).

Finally $P \subseteq \bigcup_{m=1}^{\infty} S_m$ is a subset of a countable set

$\Rightarrow P$ is countable by #20.

Q46 $\quad S = \{0,1,\_,9\}^{\mathbb{Z}^+}$

$\exists$ injection

$$\{0,1\}^{\mathbb{Z}^+} \longrightarrow \{0,1,\_,9\}^{\mathbb{Z}^+}$$

$$f \longmapsto f \text{ with codomain extended from } \{0,1\} \text{ to } \{0,1,\_,9\}$$

But $\{0,1\}^{\mathbb{Z}^+}$ is uncountable (shown in class).

$\{0,1\}^{\mathbb{Z}^+} \overset{\text{bijection}}{\updownarrow} \mathbb{P}(\mathbb{Z}^+)$

$\Rightarrow$ by (#20) $\{0,\_,9\}^{\mathbb{Z}^+}$ is uncountable.

Q47 There exist non computable functions!

on the one hand P = set of all computer programs is countable. (Q45).

Some programs output the value $f(n)$ given an input n.

these functions are called computable.

However there are uncountably many functions (Q46).

$\Rightarrow$ Some must be non-computable.

Question raised in class on 11/19/09

<u>To show:</u> $\mathbb{R}^{\mathbb{R}}$ has cardinality $2^{2^{\aleph_0}}$.

Recall the definition of $\mathbb{R}^{\mathbb{R}}$ ...

$$\mathbb{R}^{\mathbb{R}} = \{ f \mid f : \mathbb{R} \longrightarrow \mathbb{R} \text{ is a function} \}.$$

Recall also that $|\mathbb{R}| = 2^{\aleph_0}$, so that

$$|\mathbb{P}(\mathbb{R})| = 2^{|\mathbb{R}|} = 2^{2^{\aleph_0}}.$$

Therefore, the claim realy asserts the existence of a bijection from $\mathbb{R}^{\mathbb{R}}$ to $\mathbb{P}(\mathbb{R})$.

We'll show this by exhibiting two injections; one from $\mathbb{P}(\mathbb{R})$ to $\mathbb{R}^{\mathbb{R}}$, and one from $\mathbb{R}^{\mathbb{R}}$ to $\mathbb{P}(\mathbb{R})$, and then appealing to S–B theorem.

① <u>The injection $\mathbb{P}(\mathbb{R}) \longrightarrow \mathbb{R}^{\mathbb{R}}$.</u>

$$\mathbb{P}(\mathbb{R}) \xrightarrow[f_1]{} \{0,1\}^{\mathbb{R}} \xrightarrow[f_2]{} \mathbb{R}^{\mathbb{R}}$$

f_1 takes a subset $A \subseteq \mathbb{R}$ to its characteristic function, χ_A. We've seen that f_1 is a bijection (when $\mathbb{R}$ was replaced by an arbitrary set, in fact) in class notes.

f_2 takes a function $g : \mathbb{R} \to \{0,1\}$
to the function $\mathbb{R} \xrightarrow{g} \{0,1\} \xhookrightarrow{i} \mathbb{R}$
$i \circ g : \mathbb{R} \to \mathbb{R}$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[rr, bend right, "i \circ g"'] & \{0,1\} \arrow[r, hook, "i"] & \mathbb{R} \end{tikzcd}$$

(simply extends the codomain from $\{0,1\}$ to $\mathbb{R}$).

$$\begin{aligned} i : \{0,1\} &\longrightarrow \mathbb{R} \\ : 0 &\longmapsto 0 \\ : 1 &\longmapsto 1 \end{aligned}$$

<u>Exercise</u> Verify that f_2 is injective.

The composition $f_2 \circ f_1 : \mathbb{P}(\mathbb{R}) \longrightarrow \mathbb{R}^{\mathbb{R}}$ is injective.

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(2) The injection $\mathbb{R}^{\mathbb{R}} \to \mathbb{P}(\mathbb{R})$.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be a set of ordered pairs $(x, f(x))$, and so is a subset of $\mathbb{R}^2$.

$$\Rightarrow \quad \begin{aligned} j : \mathbb{R}^{\mathbb{R}} &\hookrightarrow \mathbb{P}(\mathbb{R}^2) \\ : f &\longmapsto \{(x, f(x)) \mid x \in \mathbb{R}\} \end{aligned}$$

is an inclusion.
(injective map).
verify that!

In class notes we proved that there exists a bijection
$h : \mathbb{R}^2 \to \mathbb{R}$.

This defines a bijection $H : \mathbb{P}(\mathbb{R}^2) \longrightarrow \mathbb{P}(\mathbb{R})$
$: A \longmapsto h(A)$

verify that H is bij.

image of A under h.

Thus the composition $H \circ j$

$$\begin{tikzcd} \mathbb{R}^{\mathbb{R}} \arrow[r] \arrow[rr, bend left] & \mathbb{P}(\mathbb{R}^2) \arrow[r] & \mathbb{P}(\mathbb{R}) \end{tikzcd}$$

is injective.

We have two injective maps

$$\begin{tikzcd} \mathbb{R}^{\mathbb{R}} \arrow[r, bend left, "H \circ j"] & \mathbb{P}(\mathbb{R}) \arrow[l, bend left, "f_2 \circ f_1"] \end{tikzcd}$$

The Schröder - Bernstein theorem implies that there exists a bijection between $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{P}(\mathbb{R})$.

$$\Rightarrow \quad |\mathbb{R}^{\mathbb{R}}| = |\mathbb{P}(\mathbb{R})| = 2^{2^{\aleph_0}}.$$