Fa'10: MATH 5853–001 **Topology Midterm** **Noel Brady**

Instructions. Answer as many questions as you can. Show all the steps of your work clearly.

Q1. Let $\mathcal{T}$ denote the standard topology on $\mathbb{R}$, and let $\mathcal{T}_l$ and $\mathcal{T}_u$ denote the lower limit and upper limit topologies on $\mathbb{R}$. Recall that $\mathcal{T}_l$ (resp. $\mathcal{T}_u$) has basis

$$\mathcal{B}_l = \{[a,b) \mid a,b \in \mathbb{R}, a<b\} \qquad (\text{resp. } \mathcal{B}_u = \{(a,b] \mid a,b \in \mathbb{R}, a<b\})$$

We saw in class (homework) that $\mathcal{T}_l$ is not comparable to $\mathcal{T}_u$, and that each are strictly finer than the standard topology on $\mathbb{R}$.

(a) Prove that $(\mathbb{R}, \mathcal{T}_l)$ is homeomorphic to $(\mathbb{R}, \mathcal{T}_u)$.

(b) Does there exist a homeomorphism from $\mathbb{R}$ with the standard topology to $(\mathbb{R}, \mathcal{T}_l)$?

Hint: First prove that the standard topology can be defined using a basis consisting of countably many elements. Then think about the question again.

Q2. Let Ω denote the smallest uncountable ordinal, and give the set $S_\Omega \cup \{\Omega\}$ the order topology.

(a) Prove that Ω is a limit point of the set S_Ω.

(b) Prove that there is no sequence in S_Ω which converges to Ω in the order topology.

(c) Conclude that $S_\Omega \cup \{\Omega\}$ with the order topology is not a metrizable space.

Q3. Suppose that $\mathbb{R}$ has the standard topology. Define the *box* and the *product* topologies on $\mathbb{R}^\omega$. Now, consider the function $f : \mathbb{R} \to \mathbb{R}^\omega : t \mapsto (t, t^2, t^3, t^4, \ldots)$.

(a) If $\mathbb{R}^\omega$ has the product topology, say whether or not f is continuous (give reasons for your answer).

(b) If $\mathbb{R}^\omega$ has the box topology, say whether or not f is continuous (give reasons for your answer).

Q4. Let $A \in \mathrm{SL}(2, \mathbb{Z})$. Prove that the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ induces a homeomorphism of the quotient space $\widehat{A} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.

Q5. (a) Let X and Y be topological spaces, and give $X \times Y$ and $Y \times X$ the product topologies. Is $X \times Y$ homeomorphic to $Y \times X$? Give a proof or counterexample.

(b) Let X and Y be totally ordered sets, and give $X \times Y$ and $Y \times X$ the corresponding lexicographical orders. Is $X \times Y$ in its order topology homeomorphic to $Y \times X$ in its order topology? Give a proof or counterexample.

SOLUTIONS — these will have more detail than you were expected to give on the exam.

1(a) Consider $f: \mathbb{R} \to \mathbb{R} : x \mapsto -x$

- f is a bijection since $f \circ f(x) = f(-x) = x \quad \forall x \in \mathbb{R}$
- $f : \mathbb{R}_\ell \to \mathbb{R}_u$ is continuous.

<u>pf:</u> Given $(a,b] \in \mathcal{B}_u$, $f^{-1}((a,b]) = [-b,-a) \in \mathcal{B}_\ell$

Given $U \in \mathcal{T}_u \Rightarrow U = \bigcup_\alpha B_\alpha$, $B_\alpha \in \mathcal{B}_u$

$$\Rightarrow f^{-1}(U) = f^{-1}\left(\bigcup_\alpha B_\alpha\right)$$
$$= \bigcup_\alpha f^{-1}(B_\alpha)$$
$$= \bigcup_\alpha \text{ elements of } \mathcal{B}_\ell$$
$$\in \mathcal{T}_\ell$$

$\Rightarrow f : \mathbb{R}_\ell \to \mathbb{R}_u$ is cts $\boxtimes$

- Likewise, $f^{-1} : \mathbb{R}_u \to \mathbb{R}_\ell$ is cts.

1(b) <u>Claim</u> $\left(\mathcal{B}_1 = \{(a,b) \mid a,b \in \mathbb{Q},\ a<b\} \right.$ is a countable basis for standard top on $\mathbb{R}$ $\left.\right)$ — (*)

<u>pf</u> $\mathcal{B}_1$ elts are uniquely determined by $(a,b) \in \mathbb{Q}^2$ s.t. $a<b$.
From class notes $\mathbb{Q}$ & $\mathbb{Q}^2$ countable & subsets of countable sets are countable $\Rightarrow \mathcal{B}_1$ countable.

$\mathcal{B}_1 \subset \mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R},\ a<b\}$ usual basis for standard top

$\Rightarrow$ $\boxed{\mathcal{B}_1\text{-top} \subseteq \text{standard top}}$ —(i)

Conversely given $(a,b) \in \mathcal{B}$ and $x \in (a,b)$

Then $a < x$ & $x < b$.

From analysis class ($\exists$ rational # between every pair of real #s) ②

$\exists \; r_1, r_2 \in \mathbb{Q}$ s.t. $a < r_1 < x$ and $x < r_2 < b$.

$\Rightarrow \; x \in (r_1, r_2) \subseteq (a,b) \quad \Rightarrow$ standard top $\subseteq \mathcal{B}_1$-top (ii)

(i) & (ii) $\Rightarrow$ standard top $= \mathcal{B}_1$-top & so $\mathcal{B}_1$ is a countable basis for standard top on $\mathbb{R}$.

Claim $\nexists$ homeomorphism $h: \mathbb{R}_{\text{standard}} \longrightarrow \mathbb{R}_\ell$.

Pf Suppose to the contrary that $h: \mathbb{R}_{\text{stand}} \longrightarrow \mathbb{R}_\ell$ is a homeomorphism.

Then $\mathcal{B}_1' = \{h(B) \mid B \in \mathcal{B}_1\}$ is a countable basis for $\mathbb{R}_\ell$.

[Pf: $U \in \mathcal{T}_\ell \Rightarrow h^{-1}(U)$ open in standard top ... since h cts

$\rightarrow h^{-1}(U) = \bigcup_\alpha B_\alpha$, $B_\alpha \in \mathcal{B}_1$.

$\Rightarrow U = h(h^{-1}(U)) = h\left(\bigcup_\alpha B_\alpha\right) = \bigcup_\alpha h(B_\alpha)$

is a union of elements of $\mathcal{B}_1'$].

Given $x \in \mathbb{R}$, $[x, x+1)$ is open in $\mathbb{R}_\ell \Rightarrow \exists \; B_x \in \mathcal{B}_1'$ s.t.

$x \in B_x \subseteq [x, x+1)$

$\Rightarrow \inf(B_x) = \inf([x, x+1)) = x$.

This implies, $x \neq y \Rightarrow B_x \neq B_y$ (since infs are distinct)

& so we have an injection $: \mathbb{R} \rightarrow \mathcal{B}_1'$ $\Rightarrow\Leftarrow$

$: x \longmapsto B_x$

Q2 (a)

We know
- S_Ω is uncountable
- $x \in S_\Omega \Rightarrow S_x$ is countable.

Since $S_\Omega \cup \{\Omega\}$ has least element 1 & greatest elt Ω
the order topology bases elements are of 3 types:

(1) $[1, x)$ some $x \in (S_\Omega \cup \{\Omega\}) - \{1\}$

(2) (x, y) some $x, y \in S_\Omega \cup \{\Omega\}$

(3) $(x, \Omega]$ some $x \in (S_\Omega \cup \{\Omega\}) - \{\Omega\} = S_\Omega$.

$\Omega \notin$ sets of type (1) or (2).

Given any open nbd U of Ω, $\exists\, x \in S_\Omega$ s.t.

$\Omega \in (x, \Omega] \subseteq U$.

$x \in S_\Omega \Rightarrow x < \Omega$

$\Rightarrow x+1 \leq \Omega$ where $x+1$ is the successor of x.

But $S_{x+1} = S_x \cup \{x\}$ = union of countable + finite set
= countable.

$\Rightarrow x+1 \neq \Omega \implies x+1 < \Omega,$
$\implies x+1 \in S_\Omega$

$\therefore U \cap S_\Omega \supset (x, \Omega] \cap S_\Omega \ni x+1$

& so $U \cap S_\Omega \neq \emptyset$

$\Rightarrow \Omega \in \overline{S_\Omega}$.

Note: since $\Omega \notin S_\Omega$ then $\Omega \in S_\Omega'$.

2(b) Suppose $\{a_n\}_{n\in\mathbb{Z}_+}$ is a sequence in S_Ω.

$\forall n \in \mathbb{Z}_+$, $a_n \in S_\Omega \Rightarrow S_{a_n}$ countable

$\Rightarrow S_{a_n} \cup \{a_n\}$ countable (countable + singleton)

$\Rightarrow \bigcup_{n\in\mathbb{Z}_+} S_{a_n} \cup \{a_n\}$ countable (countable union of countable)

But S_Ω uncountable $\Rightarrow \bigcup_{n\in\mathbb{Z}_+} S_{a_n} \cup \{a_n\} \subsetneqq S_\Omega$

By construction, $\bigcup_{n\in\mathbb{Z}_+} S_{a_n} \cup \{a_n\}$ is an order ideal in S_Ω.

Th^m from class $\Rightarrow \bigcup_{n\in\mathbb{Z}_+} S_{a_n} \cup \{a_n\} = (S_\Omega)_b$ for some $b \in S_\Omega$

$\Rightarrow a_n < b \quad \forall n \in \mathbb{Z}_+$

$\Rightarrow (b, \Omega]$ is open nbd of Ω which does not contain any a_n.

$\Rightarrow a_n \not\rightarrow \Omega$.

2(c) If $S_\Omega \cup \{\Omega\}$ were metrizable (induced by metric, d), and since $\Omega \in S_\Omega'$ (by 2(a)) we could pick

$$a_n \in S_\Omega \cap B_d(\Omega, \tfrac{1}{n}) \neq \emptyset$$

and argue that $\{a_n\}_{n\in\mathbb{Z}_+}$ is a sequence in S_Ω converging to Ω. This would contradict 2(b).

Q3

Basis for box topology $\rightarrow$ $\mathcal{B}_{box} = \left\{ \prod_{n=1}^{\infty} (a_n, b_n) \;\middle|\; a_n, b_n \in \mathbb{R},\ a_n < b_n \quad \forall n \in \mathbb{Z}_+ \right\}$

Basis for product topology $\rightarrow$ $\mathcal{B}_{prod} = \left\{ \prod_{n=1}^{\infty} U_n \;\middle|\; U_n = \mathbb{R} \text{ for all but finitely many } n \in \mathbb{Z}_+, \ U_n = (a_n, b_n),\ a_n < b_n \in \mathbb{R} \text{ for remaining indices} \right\}$.

(a) $f_n : \mathbb{R} \to \mathbb{R} : t \longmapsto t^n$ is continuous (Analysis class)

$\Rightarrow$ $Pr_n \circ f : \mathbb{R} \to \mathbb{R}$ is cts $\forall n \in \mathbb{Z}_+$

$\Rightarrow$ $f : \mathbb{R} \to \mathbb{R}^n_{prod}$ is cts (by thm from class)

(b) **No** Let $U = \prod_{n=1}^{\infty} \left(-\frac{1}{n^n}, \frac{1}{n^n} \right)$

$\vec{0} = (0, 0, \ldots) \in U$ & U open in box topology

$0 \in f^{-1}(U) = (-1, 1) \cap [0, \frac{1}{2}) \cap (-\frac{1}{3}, \frac{1}{3}) \cap [0, \frac{1}{4}) \cap \ldots$

$= \left(\bigcap_{j=1}^{\infty} \left(-\frac{1}{2j-1}, \frac{1}{2j-1} \right) \right) \cap \bigcap_{j=1}^{\infty} [0, \frac{1}{2j})$

$= \{0\}$ is not open in $\mathbb{R}$.

Q4 $A \in SL(2,\mathbb{Z})$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ say. $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL(2,\mathbb{Z})$.

$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (x,y) \longmapsto (ax+by,\ cx+dy)$ is cts since each coordinate function is (a sum of cts functions & hence is) cts, and $\mathbb{R}^2$ (target) has the product topology.

Likewise, $A^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \longmapsto (dx-cy,\ -bx+ay)$ is cts.

Let $q : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 : (x,y) \longmapsto q(x,y) = [(x,y)]$ denote the map which takes a point to its $\sim$ equivalence class & give $\mathbb{R}^2/\mathbb{Z}^2$ the quotient topology.

Then q is cts & the composition $q \circ A : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is cts.

Further, $(x,y) \sim (x',y')$

$\Rightarrow (x,y) - (x',y') \in \mathbb{Z}^2$

$\Rightarrow A((x,y) - (x',y')) \in \mathbb{Z}^2$ ... since A entries are $\in \mathbb{Z}$.

$\Rightarrow A(x,y) - A(x',y') \in \mathbb{Z}^2$ ---- A linear

$\Rightarrow A(x,y) \sim A(x',y')$

$\Rightarrow q \circ A(x,y) = q \circ A(x',y')$

Thm from class about quotient spaces $\Longrightarrow$ we get a well-defined, cts function $\hat{A} : \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$, defined

by setting $\hat{A}\left([(x,y)]\right) \underset{def}{=\!=} q \circ A(x,y) = [A(x,y)]$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "A"] \arrow[d, "q"'] \arrow[dr, "q \circ A"] & \mathbb{R}^2 \arrow[d, "q"] \\
\mathbb{R}^2/\mathbb{Z}^2 \arrow[r, dashed, "\hat{A}"'] & \mathbb{R}^2/\mathbb{Z}^2
\end{tikzcd}$$

$$\boxed{\hat{A}\left([(x,y)]\right) = [A(x,y)]}$$

Likewise, $\widehat{A^{-1}} : \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 : [(x,y)] \longmapsto [A^{-1}(x,y)]$

is cts.

Finally, $\widehat{A^{-1}} \circ \hat{A}\left([(x,y)]\right) = \widehat{A^{-1}}\left([A(x,y)]\right)$ ... def of $\hat{A}$

$= [A^{-1}A(x,y)]$ ... def of $\widehat{A^{-1}}$

$= [(x,y)]$, $\forall [(x,y)] \in \mathbb{R}^2/\mathbb{Z}^2$

and $\hat{A} \circ \widehat{A^{-1}}\left([(x,y)]\right) = \hat{A}\left([A^{-1}(x,y)]\right)$ ... def of $\widehat{A^{-1}}$

$= [A\,A^{-1}(x,y)]$ ... def of $\hat{A}$

$= [(x,y)]$ $\forall [(x,y)] \in \mathbb{R}^2/\mathbb{Z}^2$

Hence, $\hat{A}$, $\widehat{A^{-1}}$ are cts inverses of each other

& hence are homeomorphisms.

Q5 (a) Are homeomorphic

$$f : X \times Y \longrightarrow Y \times X \; : (x,y) \longmapsto (y,x)$$

$$g : Y \times X \longrightarrow X \times Y \; : (y,x) \longmapsto (x,y)$$

Note $\quad f \circ g = \mathbb{1}_{Y \times X} \quad \& \quad g \circ f = \mathbb{1}_{X \times Y}$

$\Rightarrow f, g$ are bijections.

f^{-1}(basis elt for product top on $Y \times X$)

$= f^{-1}(V \times U) \quad \ldots \; V \in \mathcal{T}_Y, \; U \in \mathcal{T}_X$

$= U \times V \quad =$ basis elt for prod top on $X \times Y$

$\Rightarrow f$ is cts.

Likewise g is cts.

$\leftarrow$ cts inverses of each other.

$\Rightarrow f, g$ are homeomorphisms.

Q 5 (b) Not homeomorphic.

eg $X = \mathbb{Z}_+$ with usual ordering

$Y = \{1,2\}$ with ordering inherited from $\mathbb{Z}_+$.

$X \times Y$, lex is $\mathbb{Z}_+ \times \{1,2\}$, lex which is order isomorphic to $\mathbb{Z}_+$ (class notes) & hence order top is discrete.

$Y \times X$, lex is $\{1,2\} \times \mathbb{Z}_+$, lex does not give a discrete topology.

In particular, an open neighbourhood of $(2,1)$ will have to contain an interval of the form

$((1,n), (2,2))$ for some $n \in \mathbb{Z}_+$

This contains ∞ly many points $\{(1,m) \mid m > n\}$ in addition to $(2,1)$.

$\Rightarrow \{(2,1)\}$ is not open in the lex-order topology.