Math 5853–001 Topology Final **NAME:** ________________

Q1]. . . [25 points]

1. Define what it means for a topological space to be connected.

2. Prove that the unit interval $[0, 1]$ is connected.

3. Define what it means for a topological space to be path-connected.

4. Prove that path-connectedness implies connectedness.

5. What about the converse to the previous statement? (Give proof or counterexample).

Q2]. . . [25 points]

1. Define what it means for a topological space to be compact.

2. Prove that closed subspaces of compact spaces are compact.

3. Prove that compact subspaces of Hausdorff spaces are closed.

4. Prove that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

5. What happens if we remove the compactness restriction on the domain space in part 4 above?

6. What happens if we remove the Hausdorff restriction on the range space in part 4 above?

Q3]. . . [25 points]

1. Define quotient map and quotient topology.

2. If $q : X \to Y$ is a quotient map and $g : Y \to Z$ is a map, prove that g is continuous if and only if $g \circ q$ is continuous.

3. Give a detailed proof of the fact that the quotient space obtained from $[0,1]^2$ by identifying the point $(0,t)$ with the point $(1,t)$ for each $t \in [0,1]$ is homeomorphic to the cylinder $S^1 \times [0,1] \subset \mathbb{R}^3$. Recall that $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Q4]. . . [25 points] True or False. [Supply "one phrase" proof or counterexample as necessary]

1. the continuous image of a connected space is connected.

2. every closed subspace of a locally compact space is compact.

3. the subspace $[0, 1)$ of $\mathbb{R}$ with the usual topology is homeomorphic to the subspace $[0, 1)$ of $\mathbb{R}_l$ ($\mathbb{R}$ with the lower limit topology).

4. the intervals $[0, 1)$ and $(0, 1)$ with the standard topology are homeomorphic.

5. a subset A in a topological space X is closed if and only if every sequence of points in A converges to a point of A.

6. the set of rational coordinate points $\{(q, -q) \mid q \in \mathbb{Q}\}$ in the space $\mathbb{R}_l^2$ is a closed subset.

7. products of regular spaces are regular (in product topology).

8. products of normal spaces are normal (in product topology).

9. if a space is Lindelof and regular, then it is normal.

10. if a space is second countable and regular then it is metrizable.

11. the uniform topology on a countable product of copies of $\mathbb{R}$ (each with the usual topology) is strictly finer than the product topology.

12. if two locally compact Hausdorff spaces are homeomorphic, then their one point compactifications are homeomorphic.

13. if the one-point compactifications of two locally compact Hausdorff spaces (X and Y say) are homeomorphic, then the original space X and Y are homeomorphic.

14. $\mathbb{R}^\omega$ is connected in the product topology.

15. $\mathbb{R}^\omega$ is connected in the uniform topology.