Math 1914. Extra Hwk IV Linear Approximations with Error Estimates

The Derivative as a Linear Approximation.

- We saw in class that if $f'(a)$ exists, then the following expression holds

$$f(x) \;=\; f(a) + f'(a)(x-a) + \epsilon.(x-a) \qquad (I)$$

  where $\epsilon \to 0$ as $x \to a$.

- So if $f(x)$ is differentiable at the input a the tangent line $y = f(a) + f'(a)(x-a)$ approximates $f(x)$ with an error term of $\epsilon.(x-a)$ which tends to 0 as $x \to a$ faster than $(x-a)$ tends to 0. We see this because $\epsilon.(x-a)$ is the product of two quantities that are both going to 0 as $x \to a$, and one of them is $(x-a)$.

- Earlier this semester in class, Landon asked if we could quantify this error somehow. Are there explicit bounds for the error, rather than just saying that "the error goes to zero faster than $(x-a)$?"

Linear Approximations with Error Estimates. We are given a function which is twice differentiable on an interval containing a and x. Then

$$f(x) \;=\; f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2 \qquad (II)$$

for some c between a and x.

- Note that the c depends on the input x.

- So the difference $f(x) - (f(a) + f'(a)(x-a))$ or, equivalently, the error in approximating $f(x)$ with the linear function (tangent line) $f(a) + f'(a)(x-a)$ is equal to $\frac{f''(c)}{2}(x-a)^2$. This is bounded in size by $\frac{|f''(c)|}{2}|x-a|^2$. We will explore this error bound in various cases below.

- In a later course you will see the following result (with appropriate hypotheses on f)

$$f(x) \;=\; f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

  for some c between a and x. This tells you how well the n-th degree Taylor polynomial

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

  approximates the function $f(x)$. So our result is a special case of Taylor approximations (when $n = 1$).

Q1. Exploring the Error Bounds

(a) Draw a rough sketch of the graph of $f(x) = \sqrt{x}$ on the interval $[0, 30]$.

(b) Look at the graph. Now say which linearization will give the better approximation to the function on the given interval:

- the tangent line at the point $(1, 1)$ as an approximation to f over the interval $[1, 2]$, and
- the tangent line at the point $(25, 5)$ as an approximation to f over the interval $[25, 26]$.

(c) Check your intuition above.

- Write down the linearization for $f(x) = \sqrt{x}$ at the input 1, and verify that the error $\frac{|f''(c)|}{2}|x-1|^2$ is bounded above by $\frac{1}{8}$ on the interval $[1, 2]$.
- Write down the linearization for $f(x) = \sqrt{x}$ at the input 25, and determine an upper bound for the error $\frac{|f''(c)|}{2}|x-25|^2$ on the interval $[25, 26]$.
- Which bound is smaller?

(d) What would happen if we used these linear approximations to estimate the function on smaller intervals; say on intervals of length less than 1? Then the $|x-a|^2$ term would become very small, giving us even better estimates.

- Give an upper bound for the error when we approximate $f(x)$ by the tangent line at $(1, 1)$ on the interval $[1, 1.01]$? Check your estimate by comparing the value of the appropriate linearization $L(1.01)$ with the value of the function $f(1.01)$.
- Give an upper bound for the error when we approximate $f(x)$ by the tangent line at $(25, 5)$ on the interval $[25, 25.01]$? Check your estimate by comparing the value of the appropriate linearization $L(25.01)$ with the value of the function $f(25.01)$.

Q2. Proving that expression *(II)* is true. First rewrite the expression using b in place of x. We have to show that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2 \qquad (II)$$

for some c between a and b.

We will use two applications of Rolle's Theorem to establish this.

(a) Consider the following quadratic polynomial function defined on $[a, b]$.

$$g(x) = f(a) + f'(a)(x-a) + M(x-a)^2$$

where M is some constant we have yet to determine. Check that $g(a) = f(a)$ and that $g'(a) = f'(a)$. So f and g agree (up through their first derivatives) at the endpoint a.

(b) Compute the second derivative $g''(x)$.

(c) We choose the constant M so that $g(b) = f(b)$. Now the functions f and g agree at the endpoint b. We could find out what the value of M should be now (in terms of $f(b)$, $f(a)$, $f'(a)$ and $(b-a)$ terms), but we will find another expression for M anon.

(d) Let $h(x) = f(x) - g(x)$ be the difference of the function f and the quadratic polynomial. Say why h is twice differentiable on the interval $[a, b]$.

(e) Check that $h(a) = 0 = h(b)$. What do you conclude from Rolle's Theorem?

(f) Check that $h'(a) = 0$. Combine this with the conclusion to the part above. You can apply Rolle's theorem (to the function h'). What is the conclusion of Rolle's theorem in this case.

(g) Recall that $h = f - g$. Now, use the result for g'' above, and the output of the second application of Rolle's theorem above to write M in terms of f'', and to conclude that (II) holds.