Extra Hwk II sol^ns

1(a) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ $f(4) = 2$

$f'(4) = \frac{1}{2(2)} = \frac{1}{4}$

Linearization $L(h) = f(4) + f'(4)h$

$= 2 + \frac{1}{4}h$

1(b) $\sqrt{4.001} \approx L(0.001) = 2 + \frac{0.001}{4} = 2.00025$

Compare with $\sqrt{4.001} = 2.000249984\ldots$

$\sqrt{4.0001} \approx L(0.0001) = 2 + \frac{0.0001}{4} = 2.000025$

Compare with $\sqrt{4.0001} = 2.0000249998 4\ldots$

1(c) $f(x) = \frac{1}{1+x} = (1+x)^{-1}$ $f'(x) = -(1+x)^{-2}\frac{d}{dx}(1+x)$

$= \frac{-1}{(1+x)^2}$

$f(0) = 1$ $f'(0) = \frac{-1}{1} = -1$

$L(x) = f(0) + f'(0)x$ --- Linearization at $\boxed{x=0}$

$= 1 + (-1)x$

$= 1 - x$

ie. $f(u) \approx L(u)$

$$\boxed{\frac{1}{1+u} \approx 1 - u}$$ for small values of u.

Furthermore

$$f(u) = 1 - u + \varepsilon u \quad \text{where } \varepsilon \to 0 \text{ as } u \to 0$$

by linear approx ~~formula~~ $f'(0)$.

1(d)

$$\frac{1}{1.004} = f(0.004)$$

$$\approx 1 - 0.004 = 0.996$$

Compare w/ $\frac{1}{1.004} = 0.9960159\ldots$

Also

1(e)

$$\frac{1}{3.006} = \frac{1}{3}\left(\frac{1}{1.002}\right) \approx \frac{1}{3}(1 - 0.002) = \frac{1}{3}(0.998)$$

$$= 0.33266\ldots$$

Compare with $\frac{1}{3.006} = 0.332667997\ldots$

Q2

$$f(a+h) = f(a) + f'(a)h + \varepsilon_1 h \qquad \varepsilon_1 \to 0 \text{ as } h \to 0$$

$$g(a+h) = g(a) + g'(a)h + \varepsilon_2 h \qquad \varepsilon_2 \to 0 \text{ as } h \to 0$$

$$\Rightarrow f(a+h)g(a+h) = \left(f(a) + f'(a)h + \varepsilon_1 h\right)\left(g(a) + g'(a)h + \varepsilon_2 h\right)$$

$$= f(a)g(a) + \left(f'(a)g(a) + f(a)g'(a)\right)h$$

$$+ \left(f(a)\varepsilon_2 + f'(a)g'(a)h + \varepsilon_1 g(a) + f'(a)\varepsilon_2 h + g'(a)\varepsilon_1 h + \varepsilon_1\varepsilon_2 h\right)h$$

where $\varepsilon_3 = \left(f(a)\varepsilon_2 + f'(a)g'(a)h + \varepsilon_1 g(a) + f'(a)\varepsilon_2 h + g'(a)\varepsilon_1 h + \varepsilon_1\varepsilon_2 h \right)$

$\to 0$ as $h \to 0$ (since $\varepsilon_1, \varepsilon_2, h \to 0$).

$\Rightarrow$ $f(x)g(x)$ is differentiable at a and

$$\left.\frac{d}{dx}\big(f(x)g(x)\big)\right|_{x=a} = f'(a)g(a) + f(a)g'(a)$$

Product Rule!

Q3

$$f(a+h) = f(a) + f'(a)h + \varepsilon_1 h$$
$$g(a+h) = g(a) + g'(a)h + \varepsilon_2 h$$

$\varepsilon_1, \varepsilon_2 \to 0$ as $h \to 0$

$$\Rightarrow \frac{f(a+h)}{g(a+h)} = \frac{f(a) + f'(a)h + \varepsilon_1 h}{g(a) + g'(a)h + \varepsilon_2 h}$$

$$= \frac{1}{g(a)}\left[f(a) + f'(a)h + \varepsilon_1 h\right] \frac{1}{\left[1 + \frac{g'(a)}{g(a)}h + \frac{\varepsilon_2}{g(a)}h\right]}$$

$$= \frac{1}{g(a)}\left[f(a) + f'(a)h + \varepsilon_1 h\right]\left[1 - \left(\frac{g'(a)}{g(a)}h + \frac{\varepsilon_2}{g(a)}h\right) + \varepsilon\left(\frac{g'(a)}{g(a)}h + \frac{\varepsilon_2}{g(a)}h\right)\right]$$

u

where $\varepsilon \to 0$ as $u \to 0$

... where $\varepsilon \to 0$ as $u \to 0$

$$u = \left(\frac{g'(a)}{g(a)} + \frac{\varepsilon_2}{g(a)}\right)h$$

Note $u \to 0$ as $h \to 0$

so $\varepsilon \to 0$ as $h \to 0$.

Therefore

$$\frac{f(a+h)}{g(a+h)} = \frac{1}{g(a)}\Bigg[f(a) - \frac{f(a)\, g'(a)}{g(a)} h - \frac{f(a)\, \varepsilon_2\, h}{g(a)}$$

$$+ f(a) \cdot \varepsilon \cdot \left(\frac{g'(a)}{g(a)} h + \frac{\varepsilon_2 h}{g(a)}\right)$$

$$+ f'(a) h$$

$$+ f'(a) h \left[(\varepsilon - 1)\left(\frac{g'(a)}{g(a)} + \frac{\varepsilon_2}{g(a)}\right) h\right]$$

$$+ \varepsilon_1 h \left[1 + (\varepsilon - 1)\left(\frac{g'(a)}{g(a)} + \frac{\varepsilon_2}{g(a)}\right) h\right]\Bigg]$$

Cleaning up.

$$\frac{f(a+h)}{g(a+h)} = \frac{f(a)}{g(a)} + \frac{f'(a) g(a) - f(a)\, g'(a)}{(g(a))^2} h$$

$$+ \frac{1}{g(a)}\left[- \frac{f(a)\, \varepsilon_2 h}{g(a)} + f(a)\varepsilon\left(\frac{g'(a)}{g(a)} + \frac{\varepsilon_2}{g(a)}\right)h + \qquad \right]$$

OR

$$\frac{f(a+h)}{g(a+h)} = \frac{f(a)}{g(a)} + \left(\frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2} \right) h$$

$$+ \underbrace{\frac{1}{(g(a))^2} \Big[-f(a)\varepsilon_2 + f(a)\left(g'(a)+\varepsilon_2\right)\varepsilon + f'(a)(\varepsilon-1)\left(g'(a)+\varepsilon_2\right)h + \varepsilon_1 \left[1 + (\varepsilon-1)\left(g'(a)+\varepsilon_2\right)h \right] \Big]}_{=\ \varepsilon_3} h$$

$\varepsilon_3 \to 0$ as $h \to 0$ since it is a sum of 4 terms, each $\to 0$ as $h \to 0$

$$\left.\begin{array}{l} 1\text{st} = \text{multiple of } \varepsilon_2 \\ 2\text{nd} = \text{multiple of } \varepsilon \\ 3\text{rd} = \text{multiple of } h \\ 4\text{th} = \text{multiple of } \varepsilon_1 \end{array}\right) \to 0 \text{ as } h \to 0.$$

$\Rightarrow \dfrac{f(x)}{g(x)}$ is differentiable at $x=a$ and

$$\boxed{\left.\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\right|_{x=a} = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}}$$

Extra Hwk III (Sol'ns)

Step ①

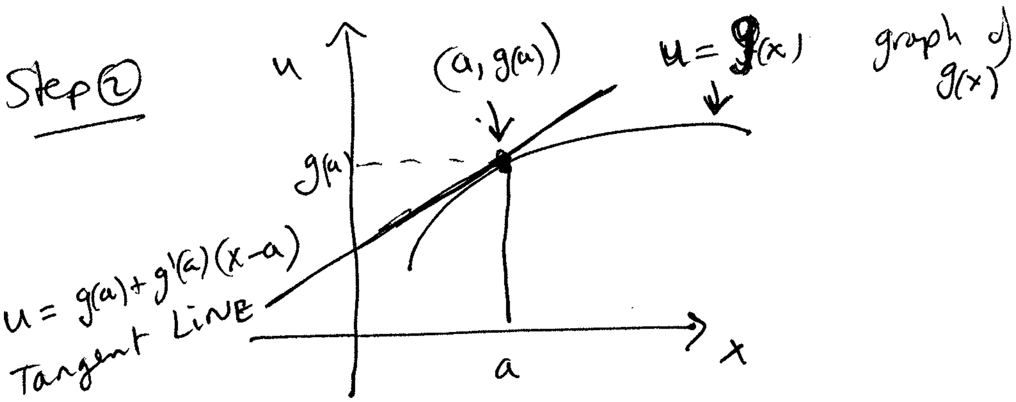
$y = l_1(u) = 2u + 3$

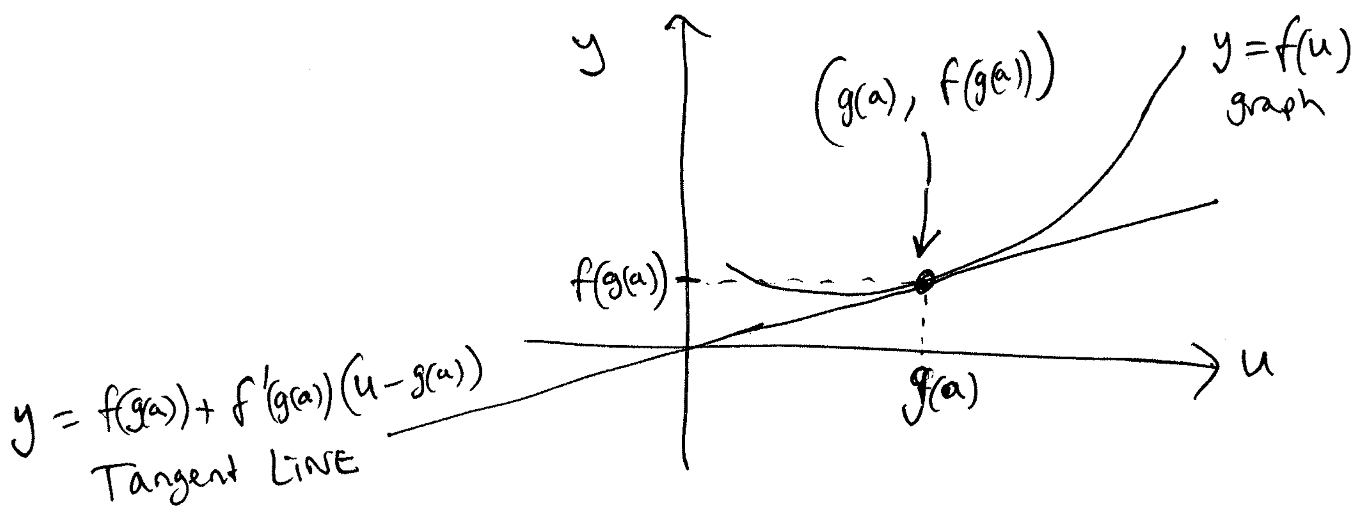
$u = l_2(x) = 5x - 4$

$$y = (l_1 \circ l_2)(x) = 2(5x-4) + 3$$
$$= 10x - 8 + 3$$
$$= 10x - 5$$

slope of composite line $= 10$ = product of slopes of original 2 lines l_1 & l_2.

Step ②





Step(3) comp. of tangent lines is

$$y = f(g(a)) + f'(g(a))\left[\left(\cancel{g(a)} + g'(a)(x-a)\right) - \cancel{g(a)}\right]$$

$$= f(g(a)) + f'(g(a))\, g'(a)\, (x-a)$$

is a straight line with slope $= f'(g(a))\, g'(a)$.

Step(4) $u = g(x) = g(a) + g'(a)(x-a) + \varepsilon_1 (x-a)$

where $\varepsilon_1 \to 0$ as $x \to a$.

$\boxed{\text{Note as } x \to a \text{ then } u \to g(a) + g'(a).0 + (0)(0) = g(a)}$

$$y = f(u) = f(g(a)) + f'(g(a))(u - g(a)) + \varepsilon_2 (u - g(a))$$

where $\varepsilon_2 \to 0$ as $u \to g(a)$

$$y = f(g(x)) = f(g(a)) + f'(g(a))\left(\cancel{g(a)} + g'(a)(x-a) + \varepsilon_1 (x-a) - \cancel{g(a)}\right)$$

$$+ \varepsilon_2 \left(\cancel{g(a)} + g'(a)(x-a) + \varepsilon_1 (x-a) - \cancel{g(a)}\right)$$

$$y = f(g(x)) = f(g(a))$$

$$+ f'(g(a)) \cdot g'(a) \cdot (x-a)$$

$$+ \underbrace{\left[f'(g(a))\, \varepsilon_1 + \varepsilon_2\, g'(a) + \varepsilon_1 \varepsilon_2 \right]}_{\varepsilon_3} (x-a)$$

Note as $x \to a$ then $\varepsilon_1, \varepsilon_2 \to 0$

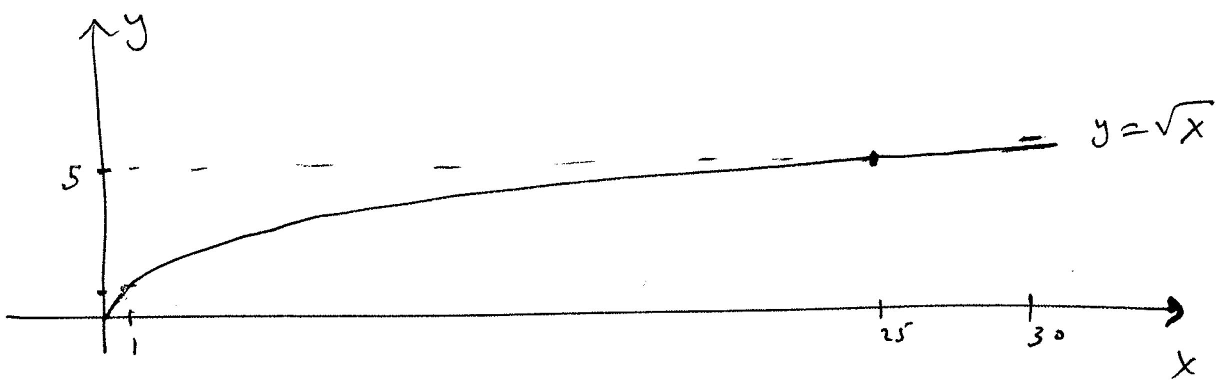
$\Rightarrow \varepsilon_3 \to 0$

Therefore, $f(g(x))$ is differentiable at $x = a$, and

$$\left. \frac{d}{dx}\big(f(g(x))\big) \right|_{x=a} = f'(g(a)) \cdot g'(a)$$

Ch. Rule. $\boxed{//}$

Extra Hwk IV (Sol'ns)



Graph is "straighter" near (25,5) than near (1,1)

Expect linearization at 25 to be better approx than linearization at 1.

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ $\qquad f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2}$

$f(1) = \sqrt{1} = 1$ $\qquad f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

$1 \le c \le 2 \Rightarrow \underbrace{1^{3/2} \le c^{3/2}}_{\Downarrow} \le 2^{3/2} = 2\sqrt{2}$

$\Rightarrow \frac{1}{c^{3/2}} \le \frac{1}{1^{3/2}} = 1$

$\Rightarrow \frac{1}{4c^{3/2}} \le \frac{1}{4}$ —(*)

$\sqrt{x} = 1 + \frac{1}{2}(x-1) + \frac{f''(c)}{2}(x-1)^2$ for some c in [1, x]

Now c in $[1,x]$ & x in $[1,2]$ $\Rightarrow$ c in $[1,2]$

$$\Rightarrow \quad |f''(c)| = \frac{1}{4c^{3/2}} \le \frac{1}{4} \quad \text{by } (*).$$

So error term is $\frac{|f''(c)|}{2}|x-1|^2$

$$\le \frac{\frac{1}{4}}{2}(1)^2 = \boxed{\frac{1}{8}}$$

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if c is in $[25, x]$ & x is in $(25, 26]$ then

c is in $[25, 26]$ and

$$25 \le c \le 26$$

$$25^{3/2} \le c^{3/2} \le 26^{3/2}$$

$$\Rightarrow \quad \frac{1}{c^{3/2}} \le \frac{1}{125}$$

$$\Rightarrow \quad \frac{1}{4c^{3/2}} \le \frac{1}{500}$$

$$\sqrt{x} = f(25) + f'(25)(x-25) + \frac{f''(c)}{2}(x-25)^2$$

$$= 5 + \frac{1}{10}(x-25) + \frac{f''(c)}{2}(x-25)^2$$

So the linearization expression $5 + \frac{1}{10}(x-25)$ approximates the function $\sqrt{x}$ on the interval $[25, 26]$ with error bounded above by

$$\left|\frac{f''(c)}{2}(x-25)^2\right| \le \frac{\frac{1}{4c^{3/2}}}{2}|x-25|^2 < \frac{\frac{1}{500}}{2}(1)^2$$

$$\le \frac{1}{1{,}000}$$

This bound is much smaller than $\frac{1}{8}$ $\Rightarrow$ agrees with our intuition about the graph of $y = \sqrt{x}$.

Smaller Intervals. $[1, 1.01] \Rightarrow |x-1|^2 \le (0.01)^2$
$= 0.0001$

$\sqrt{x}$ is approximated by $1 + \frac{1}{2}(x-1)$ with error bounded by $\frac{1}{8}(0.01)^2 = \frac{0.0001}{8} = 0.0000125$

Likewise, $\sqrt{x}$ is approximated by $5 + \frac{1}{10}(x-25)$ on the interval $[25, 25.01]$ with error bounded by

$$\frac{1}{1{,}000}(0.01)^2 = 0.0000001$$

Check (1) $L(1.01) = 1 + \frac{1}{2}(1.01-1)$

$= 1.005$

$\sqrt{1.01} = 1.00498756\ldots$

difference is $< 0.000012437887 91$

< 0.0000125 (error estimate)

Check (2) $L(25.01) = 5 + \frac{1}{10}(25.01-25)$

$= 5 + \frac{1}{10}(0.01)$

$= 5.001$

$\sqrt{25.01} = 5.00099990001 9995\ldots$

difference is $< 0.0000000 09998$

< 0.0000001 (error estimate)
