Applications of S-B (Schröder-Bernstein Theorem)

App 1

Prop $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.

Pf Define $f : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$

$: n \longmapsto (n, 1)$

$f(n) = f(m) \Rightarrow (n,1) = (m,1)$

$\Rightarrow n = m$ --- defⁿ of equality of ordered pairs.

$\Rightarrow f$ injective.

Define $g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$: (n, m) \longmapsto 3^n 5^m$

$g(n,m) = g(a,b) \Rightarrow 3^n 5^m = 3^a 5^b$

$\Rightarrow n = a,\ m = b$ ... by uniqueness in F.T.A.

$\Rightarrow (n,m) = (a,b)$ ... by defⁿ of equality of ordered pairs.

$\Rightarrow g$ injective.

S-B theorem $\Rightarrow \exists$ bijection $\mathbb{N} \approx \mathbb{N} \times \mathbb{N}$.

App 2

Th$^{\underline{m}}$ $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$

Proof We have seen before that

$$\mathcal{P}(\mathbb{N}) \approx S = \{\text{infinite binary strings}\}$$
$$= \{a_1 a_2 a_3 \dots \mid a_i \in \{0,1\}, \forall i \in \mathbb{N}\}$$

By the Schröder - Bernstein Th$^{\underline{m}}$ it suffices to produce two injective functions:

$$f : S \longrightarrow \mathbb{R}$$
$$g : \mathbb{R} \longrightarrow S$$

$\boxed{f}$

$$f : S \longrightarrow \mathbb{R}$$
$$: a_1 a_2 a_3 \dots \longmapsto 0.5\, a_1 a_2 a_3 \dots$$

$$f(a_1 a_2 \dots) = g(b_1 b_2 \dots)$$

$$\Rightarrow 0.5\, a_1 a_2 \dots = 0.5\, b_1 b_2 \dots$$

$$\Rightarrow a_1 = b_1,\ a_2 = b_2,\ \dots, a_i = b_i\ \forall i \in \mathbb{N}.$$

This is because the only ambiguity with decimal expansions of real numbers is when one of the decimals ends with an ∞ string of 9's. This is not the case here.

$$\Rightarrow a_1 a_2 \dots = b_1 b_2 \dots$$

$$\Rightarrow f \text{ injective!}$$

$\boxed{9}$ We "encode" decimal expansions of real numbers using binary strings as follows.

a_j = digit $\longleftrightarrow$ encode by (a_j+1) 1's

place between digits $\longleftrightarrow$ encode by $\emptyset$

decimal point $\longleftrightarrow$ encode by $\emptyset\emptyset$.

Fact: Each real number has a unique ∞ decimal number representation (possibly ending in ∞ string of $\emptyset$'s), which does <u>not</u> end in ∞ seq. of 9's.

eg

$3.14159\ldots \longleftrightarrow 1111\,\emptyset\emptyset\,11\,\emptyset\,11111\,\emptyset\,11\,\emptyset\,111111\,\emptyset\,1\ldots$

$32 = 32.000\ldots \longleftrightarrow 1111\,\emptyset\,111\,\emptyset\emptyset\,1\,\emptyset\,1\,\emptyset\,1\,\emptyset\ldots$

$0.306 \longleftrightarrow 1\,\emptyset\emptyset\,1111\,\emptyset\,1\,\emptyset\,1111111\,\emptyset\,1\,\emptyset\,1\,\emptyset\,1\,\emptyset\ldots$

$0.36 \longleftrightarrow 1\,\emptyset\emptyset\,1111\,\emptyset\,1111111\,\emptyset\,1\,\emptyset\,1\,\emptyset\,1\,\emptyset\ldots$

$a_n \ldots a_0 \cdot a_{-1}\, a_{-2} \ldots \longleftrightarrow$

$$\underbrace{(1 \text{---} 1)}_{(a_n+1)}\emptyset\,\underbrace{(1 \text{---} 1)}_{(a_{n-1}+1)}\emptyset \ldots \underbrace{(1 \text{---} 1)}_{(a_0+1)}\emptyset\emptyset\,\underbrace{(1 \text{---} 1)}_{(a_{-1}+1)}\emptyset \ldots$$

$g : \mathbb{R} \longrightarrow S$

$\quad : x \longmapsto$ encoded version of x.

By construction / definition of the encoding

$$g(x) = g(y) \implies x \text{ \& } y \text{ have exactly same}$$

digits before + after the decimal point.

$$\implies x = y$$

$$\implies g \text{ injective.}$$

S-B theorem $\implies$ $S \approx \mathbb{R}$

$\boxed{\checkmark}$

App. 3

Prop $\quad (0,1) \approx (0,1) \times (0,1)$

where $(0,1)$ is the interval $0 < x < 1$ on the real line.

Proof

$$f : (0,1) \longrightarrow (0,1) \times (0,1)$$
$$: x \longmapsto (x, \tfrac{1}{2})$$

$$f(x) = f(y) \implies (x, \tfrac{1}{2}) = (y, \tfrac{1}{2})$$

$$\implies x = y$$ --- def^{n} of equality of ordered pairs

$$\implies f \text{ injective.}$$

$g : (0,1) \times (0,1) \longrightarrow (0,1)$

$: (0.a_1a_2a_3 \ldots, 0.b_1b_2b_3 \ldots) \longmapsto 0.a_1b_1a_2b_2 \ldots$

where each number $x \in (0,1)$ has a unique decimal expansion which does not end in an ∞ string of 9's.

$(0.a_1a_2\ldots, 0.b_1b_2\ldots)$ is the ordered pair of these unique decimal expansions for a given pair of real numbers $(x,y) \in (0,1) \times (0,1)$.

$$g(0.a_1a_2\ldots, 0.b_1b_2\ldots) = g(0.c_1c_2\ldots, 0.d_1d_2\ldots)$$

$$\Rightarrow \quad 0.a_1b_1a_2b_2\ldots = 0.c_1d_1c_2d_2\ldots$$

Neither of those ends in string of 9's

Uniqueness of decimal representation $\Rightarrow$

$$\begin{array}{ccc} a_1 = c_1 & \& & b_1 = d_1 \\ a_2 = c_2 & \& & b_2 = d_2 \\ \vdots & \vdots & \vdots \end{array}$$

ie $(0.a_1a_2\ldots, 0.b_1b_2\ldots) = (0.c_1c_2\ldots, 0.d_1d_2\ldots)$

$\Rightarrow$ g is injective.

Finally, S-B $\Rightarrow$ $\exists$ bijection $(0,1) \approx (0,1) \times (0,1)$ ∎