Prop ① $f : A \to B$ and $g : B \to C$ both injective.

Then $g \circ f : A \to C$ is injective.

Proof We have to prove

$$(\forall a_1 \in A)\ (\forall a_2 \in A)\Big(\big((g \circ f)(a_1) = (g \circ f)(a_2)\big) \longrightarrow (a_1 = a_2)\Big).$$

Suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$.

By def^{n} of composition, $g \circ f$, this means

$$g(f(a_1)) = g(f(a_2)).$$

But g is injective (by hypothesis), and so we conclude that $f(a_1) = f(a_2)$ (these are elements of B).

But f is injective (by hypothesis), and so we conclude that $a_1 = a_2$.

This is what we wanted to show.
Therefore $g \circ f$ is injective. ☒

Question ① Suppose $f : A \to B$ and $g : B \to C$ are functions, and you are told that
$g \circ f$ is injective.

What can you conclude about f, g?

$\longrightarrow$ Ans

We can conclude that f is injective.

That is

$$(g \circ f \text{ injective}) \longrightarrow (f \text{ injective}).$$

We'll prove this by establishing the <u>contrapositive</u>

$$(f \text{ not injective}) \longrightarrow ((g \circ f) \text{ not injective}).$$

<u>Proof</u> Suppose f is <u>not</u> injective.

This means $\exists\ a_1 \neq a_2$, elements of A

so that $f(a_1) = f(a_2)$.

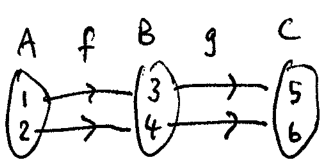
Then $g(f(a_1)) = g(f(a_2))$ ... since the inputs $f(a_1) = f(a_2)$ agree.

i.e. $(g \circ f)(a_1) = (g \circ f)(a_2)$

This means $(g \circ f)$ is <u>not</u> injective. $\boxtimes$

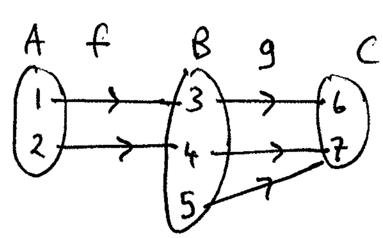
We can't make any conclusions about g

<u>eg</u> (1)



all 3 functions $(f, g, g \circ f)$ injective.

<u>eg</u> (2)



f and $g \circ f$ both injective <u>but</u> g is <u>not</u> injective,

Prop (2) If $f: A \to B$ and $g: B \to C$ are both surjective,

Then $g \circ f$ is surjective.

Proof: We have to show

$$(\forall c \in C)(\exists a \in A)\big((g \circ f)(a) = c\big).$$

Given any $c \in C$, because g is surjective (hypothesis), we know $\exists\, b \in B$ so that $g(b) = c$.

Given that $b \in B$, because f is surjective (hypothesis), we know $\exists\, a \in A$ so that $f(a) = b$.

$$\text{Now } (g \circ f)(a) = g(f(a)) \quad \text{--- def}^{n} \text{ of } g \circ f$$
$$= g(b)$$
$$= c$$

That is, we found $a \in A$ so that $(g \circ f)(a) = c$.

Thus $(g \circ f)$ is surjective. ▨

Qn (2) Suppose $f: A \to B$ and $g: B \to C$ are functions and you are told that

$g \circ f$ is surjective.

What can you conclude about f, g?

ANS. $\longrightarrow$

We can conclude that g is surjective.

<u>Proof</u> Given <u>any</u> $c \in C$.

Since $g \circ f$ is surjective (hypothesis)

we know $\exists\, a \in A$ so that

$$(g \circ f)(a) = c.$$

This means $g(f(a)) = c$.

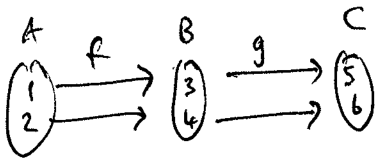
But $f(a) \in B$ and $g(f(a)) = c$

Means that c is the image of some element of B

under the map g. Thus g is surjective. $\blacksquare$

We can't conclude anything about the map f.

eg(1)

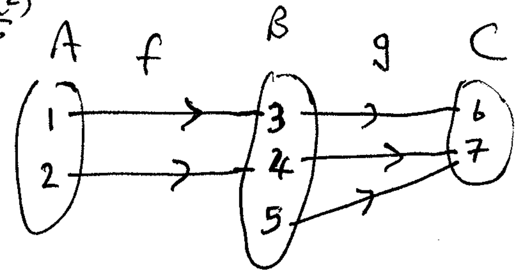


All 3 maps

f, g, $g \circ f$

are surjective

<u>eg(2)</u>



Note g and $g \circ f$ are both <u>surjective</u>

<u>but</u> f is not!

Prop 3 If $f: A \to B$ and $g: B \to C$ are bijective, then $g \circ f: A \to C$ is bijective.

Proof: By hypothesis, f and g are bijective.
In particular, f, g are injective

Prop 1 $\Rightarrow$ $g \circ f$ is injective ——— I

In particular, f and g are surjective.

Prop 2 $\Rightarrow$ $g \circ f$ is surjective ——— II

I & II $\Rightarrow$ $g \circ f$ is bijective. $\boxtimes$

Q(3) Suppose $f: A \to B$ and $g: B \to C$ are functions and you are told that
$g \circ f: A \to C$ is ~~a~~ bijection.
What can you conclude about f, g?

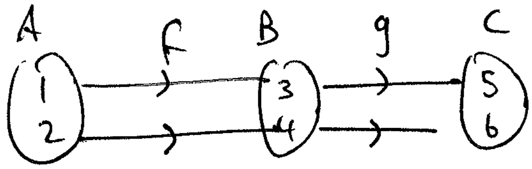
Ans from Q 1 we can conclude that f is injective.

& from Q2 we can conclude that g is surjective.

We can't conclude anything else about f, g.

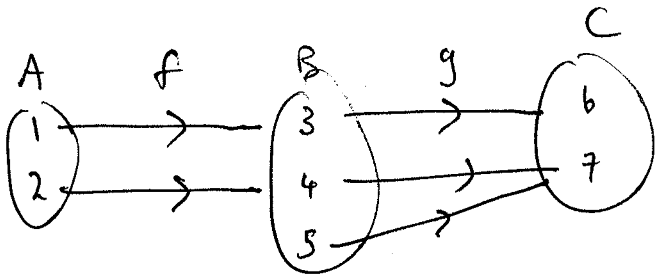
$\longrightarrow$ egs

eg ①



f, g, $g \circ f$ all bijections.

eg ②



$g \circ f$ is bijective

but

f is not (not surjective)

& g is not. (not injective).