Any postage amount of 8¢ or higher can be obtained using 5¢ & 3¢ stamps. ①

<u>Proof</u>: Use strong induction.

<u>Base Cases</u>: $P(8)$ true. $8 = 5 + 3$

$P(9)$ true. $9 = 3 + 3 + 3$

$P(10)$ true. $10 = 5 + 5$

<u>Inductive Step</u>: Assume $(P(8) \text{ true}) \wedge \dots \wedge (P(k) \text{ true})$ for $k \geq 10$.

Then $(k+1) = (k-2) + 3$ --- Note $k \geq 10$

$\Rightarrow k-2 \geq 8$

so $k-2 \in \{8, 9, \dots, k\}$

$\Rightarrow P(k-2)$ true.

$\Rightarrow$ $k-2$ can be obtained using 3¢ & 5¢ stamps

$\Rightarrow$ $(k+1)$ can be obtained using same 3¢ & 5¢ stamps, <u>and</u> an extra 3¢ stamp.

$\Rightarrow P(k+1)$ true.

<u>Conclusion</u> $P(n)$ is true $\forall n \in \mathbb{N}$, $n \geq 8$.

$\boxtimes$

Any postage amount of 12 ¢ or higher can be obtained using ~~5~~ 4 ¢ & 5 ¢ stamps. ②

Proof Use Strong Induction.

Base Cases

$P(12)$ true. $12 = 4 + 4 + 4$

$P(13)$ true. $13 = 4 + 4 + 5$

$P(14)$ true. $14 = 4 + 5 + 5$

$P(15)$ true. $15 = 5 + 5 + 5$.

Inductive step Assume $(P(12) \text{ true}) \wedge \cdots \wedge (P(k) \text{ true})$ for $k \geq 15$.

Then $(k+1) = (k-3) + 4$

$k \geq 15$

$\Rightarrow k - 3 \geq 15 - 3 = 12$

$\Rightarrow k-3 \in \{12, 13, \ldots, k\}$

$\Rightarrow P(k-3)$ is true by induction assumption.

$\Rightarrow (k-3)$ can be obtained using 4¢ & 5¢ stamps

$\Rightarrow (k+1)$ can be obtained using the same 4¢ & 5¢ stamps as above, plus an extra 4¢ stamp.

$\Rightarrow$ $P(k+1)$ true.

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 12$.

Prop Every positive integer is a sum of distinct powers of 2. ($2^0=1$, $2^1=2$, $2^2=4$, $2^3=8$, ...)

Proof: Use strong induction.

$P(n)$: n is a sum of distinct powers of 2.

Base Case $P(1)$ is true $1 = 2^0$.

Induction Step Assume $(P(1) \text{ true}) \wedge \cdots \wedge (P(k) \text{ true})$.

Given $k+1$ there are 2 cases

Case: $k+1$ is even $\Rightarrow k+1 = 2p$ some $p \in \mathbb{N}$

$1 \leq p = \frac{k+1}{2} < k+1$

$\Rightarrow p \in \{1, \ldots, k\}$

$\Rightarrow P(p)$ is true

$p = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r}$ where the exponents $a_i \in \mathbb{Z}_{\geq 0}$ are all distinct

$\Rightarrow (k+1) = 2p = 2^{a_1+1} + \cdots + 2^{a_r+1}$

$a_1+1, \ldots, a_r+1$ are still distinct!

Case: $(k+1)$ is odd. $\Rightarrow k+1 = 2p+1$ some $p \in \mathbb{N}$.

$1 \leq p = \frac{k}{2} < k+1$

$\Rightarrow p \in \{1, \ldots, k\}$

$\Rightarrow P(p)$ true

$\Rightarrow p = 2^{a_1} + \cdots + 2^{a_r}$ for distinct exponents $a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}$.

$$\Rightarrow k+1 = 2p+1 = 2\left(2^{a_1} + \cdots + 2^{a_r}\right) + 1$$

$$= 2^{a_1+1} + \cdots + 2^{a_r+1} + 2^0$$

Note the $a_1+1, \text{—}, a_r+1$ are all distinct (since the $a_1, \text{—}, a_r$ were)

and they are all ≥ 1 & so are distinct from the final exponent, 0.

$\Rightarrow P(k+1)$ true.

<u>Conclusion</u> $P(n)$ is true, for all $n \in \mathbb{N}$.

$\boxtimes$

$\sqrt{2}$ is irrational.

Proof Using strong induction. Consider the predicates:

$$P(n) : \quad \sqrt{2} \neq \frac{n}{b} \quad \text{for any } b \in \mathbb{N}.$$

Base case. $P(1)$ is true, because

$$b \in \mathbb{N} \Rightarrow b \geq 1 \Rightarrow \frac{1}{b} \leq 1 \Rightarrow \left(\frac{1}{b}\right)^2 \leq 1^2 = 1 < 2$$

so $\sqrt{2} \neq \frac{1}{b}$ for any $b \in \mathbb{N}$.

Induction step. Assume $(P(1) \text{ true}) \wedge \cdots \wedge (P(k) \text{ true})$.

We will prove (using the above) that $P(k+1)$ is true; i.e.,

$$\sqrt{2} \neq \frac{k+1}{b} \quad \text{for any } b \in \mathbb{N}.$$

Proof by contradiction. Suppose $\sqrt{2} = \frac{k+1}{b}$ for some $b \in \mathbb{N}$.

then $2 = \frac{(k+1)^2}{b^2}$ or $2b^2 = (k+1)^2$.

$\Rightarrow (k+1)^2$ is even $\Rightarrow (k+1)$ is even $\Rightarrow (k+1) = 2p$ for some $p \in \mathbb{N}$. $\Rightarrow 2b^2 = (2p)^2 = 4p^2$

$\Rightarrow b^2 = 2p^2$

$\Rightarrow b^2$ even $\Rightarrow b$ even $\Rightarrow b = 2q$ for some $q \in \mathbb{N}$.

Thus $\sqrt{2} = \frac{k+1}{b} = \frac{2p}{2q} = \frac{p}{q}$ $(*)$, $q \in \mathbb{N}$ & $p \in \mathbb{N}$

$p \leq \frac{k+1}{2} < k+1$

But $(P(1) \text{ true}) \wedge \cdots \wedge (P(k) \text{ true}) \Rightarrow$ In particular $P(p)$ true. ie. $p \in \{1, \ldots, k\}$

That means $\sqrt{2} \neq \frac{p}{b}$ for any $b \in \mathbb{N}$.

This contradicts (*)

Thus the claim that $\sqrt{2} = \frac{k+1}{b}$ for some $b \in \mathbb{N}$

led to a contradiction.

$\Rightarrow \sqrt{2} \neq \frac{k+1}{b}$ for any $b \in \mathbb{N}$.

i.e. $P(k+1)$ is true.

We've shown

- $P(1)$ true
- $(P(1) \text{ true}) \wedge \dots \wedge (P(k) \text{ true}) \longrightarrow P(k+1)$ true.

By strong induction, $P(n)$ is true $\forall n \in \mathbb{N}$.

$\Rightarrow \sqrt{2} \neq \frac{a}{b}$ for any $a, b \in \mathbb{N}$

i.e. $\sqrt{2}$ is irrational.

Def. An integer greater than 1 is said to be a prime number if its only positive divisors are itself and 1.

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eg $2, 3, 5, 7, 11, \ldots$ are primes

$4, 6, 8$ are not primes, but they are products of primes: $4 = (2)(2)$, $6 = (2)(3)$, $8 = (2)(2)(2)$.

Prop. Every integer greater than 1 is either a prime or is a product of primes.

Proof We use strong induction.

Let $P(n)$ be the sentence "n is prime or is a product of primes." We will show $P(n)$ is true for all integers $n \geq 2$.

BASE CASE: $P(2)$ is true.

2 is a prime.

Inductive Step: Assume $(P(2)\text{ true}) \wedge \cdots \wedge (P(k)\text{ true})$.

Now consider the integer $(k+1)$.

Either $(k+1)$ is prime, in which case $P(k+1)$ is true,

or $(k+1)$ is not prime.

Thus $(k+1) = ab$ where a, b are positive integers, neither of which is 1 or $(k+1)$.

Thus $2 \le a, b \le k$, and so the induction assumption holds for a & b.

That is, a is prime or is a product $p_1 — p_r$ of primes, and

b is prime or is a product $q_1 — q_s$ of primes.

Thus $(k+1) = a.b = (p_1 — p_r)(q_1 — q_s)$ is a product of primes.

(Note: $r=1$ if a prime
$s=1$ if b prime)

Therefore $P(k+1)$ is true.

By the principle of strong induction, $P(n)$ is true for all integers $n \geq 2$.

$\boxed{\text{Y}/\!/}$