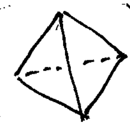
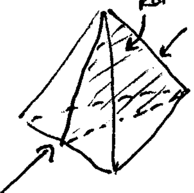
$\text{Symm}\Big($  $\Big)$  has $\begin{cases} 4 \text{ vertices} \\ 6 \text{ edges} \\ 4 \text{ faces} \end{cases}$. ①

Type ①

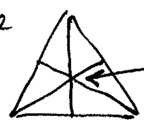
__Reflections in planes__ which (i) contain one edge of , and (ii) contain the midpoint of the opposite edge of 



There are $\boxed{6}$ of these; one per edge.

Type ②

__Rotations about lines (axes)__ which (i) contain one vertex, and (ii) the "center" of the opposite face.



Rotation angles are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$

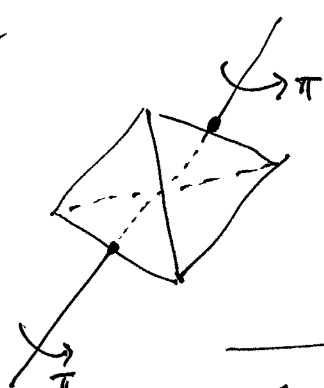
$\Rightarrow$ two per axis; one axis per vertex

$\Rightarrow \quad 2 \times 4 = \boxed{8}$ of these.

Type ③

__Rotations about lines (axes)__ which contain (i) the midpoint of one edge, and (ii) the midpoint of the opposite edge.

Rotation angle $= \pi$



One per pair of opposite edges.

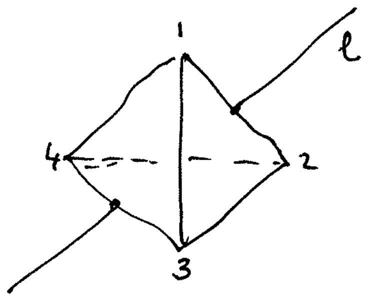
There are $\frac{6}{2} = 3$ pairs of opposite edges

$\Rightarrow \boxed{3}$ of these.

Type ④

The identity. $\boxed{1}$ of those!

Finally, we get 6 more symmetries by combining type 1 & type 3 symmetries as shown!



New symmetry is a composition

(type 1) $\circ$ (type 3)

(Reflection in plane containing 1, 3 and midpoint of [2,4]) $\circ$ (Rotation by π about axis ℓ)

Type 5

OR plane containing 2, 4 and midpoint of the line segment [1,3].

2 possibilities for each type 3 symmetry

$\Rightarrow 2 \times 3 = \boxed{6}$ of these.

Total $= 6 + 8 + 3 + 1 + 6 = \boxed{24}$

We know they are distinct and that there are no more symmetries, because they give distinct permutations of the 4 vertices $\{1,2,3,4\}$ and there are only $4! = 24$ such permutations.

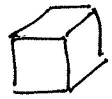
Type 1 $\longleftrightarrow$ (12) etc.. Transpositions

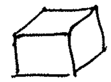
Type 2 $\longleftrightarrow$ (123) etc 3-cycles

Type 3 $\longleftrightarrow$ (12)(34) etc products of complimentary transpositions

Type 4 $\longleftrightarrow$ $\mathbb{1}$ identity permutation

Type 5 $\longleftrightarrow$ (1234) the 4-cycles

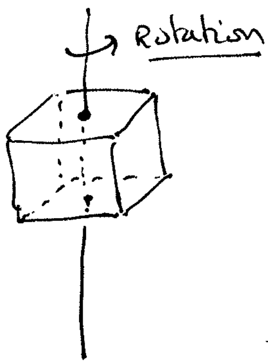
Symm ()

 has $\begin{cases} 8 \text{ vertices} \\ 12 \text{ edges} \\ 6 \text{ faces} \end{cases}$

We'll describe the symmetries which preserve "right-handedness" first of all. These are rotations about lines in 3-d.

Type ① Rotations about lines through centers of opposite faces.

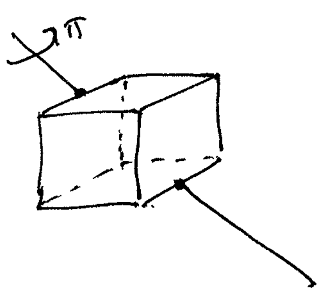
— angles are $\pi/2$, π, $\frac{3\pi}{2}$ $\Rightarrow$ 3 per line



There are $\frac{6}{2} = 3$ lines $\Rightarrow$ a total of

$3 \times 3 = \boxed{9}$ such symmetries.

Type ② Rotations about lines through centers (midpoints) of opposite edges. Angle must be π. $\Rightarrow$ 1 per line.

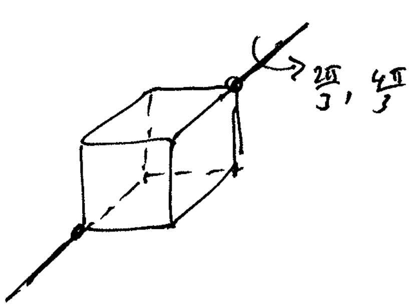


Total of $\frac{12}{2} = 6$ lines.

$\Rightarrow \boxed{6}$ of these.

Type ③ Rotations about lines through opposite vertices.

Angles are $\frac{2\pi}{3}$ & $\frac{4\pi}{3}$. $\Rightarrow$ 2 per line.



There are $\frac{8}{2} = 4$ such lines $\Rightarrow$ Total of

$2 \times 4 = \boxed{8}$ of these.

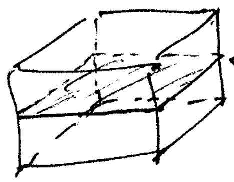
Type ④ Any of the above with 0 angle of rotation!

The identity! $\boxed{1}$ of these.

There is a total of $9+6+8+1 = 24$ rotational symmetries.

These preserve "right-handedness". (Rotate a right hand $\Rightarrow$ still get a right hand.)

Take your favorite "reflection in a plane" symmetry; e.g.,



H = mid plane between two opposite faces.

Reflection in H takes "right hand" into mirror image "left hand".

$$L_H : \mathrm{Symm}(\;) \longrightarrow \mathrm{Symm}(\;)$$

$$: \quad g \longmapsto Hg \qquad = \text{composition of } H \text{ and } g.$$

We saw in class notes that L_H is a bijection. $\Rightarrow$

$L_H(\{\text{symmetries of type } ①, ②, ③, ④\})$ is a set of 24 symmetries. (injectivity of L_H)

They are all distinct from the original 24 symmetries because they all take a right hand into its mirror image left hand.

$\Rightarrow$ we have $24 + 24 = 48$ distinct symmetries.

We were told in class that 48 was the number so we have given a description of them all.

Type ①, $\longrightarrow$ Type ④ & $H \circ (\text{Type } ①), \ldots, H \circ \text{Type } ④$.

You might like to think about giving explicit geometric descriptions of the 24 symmetries which take right hands to left hands.
