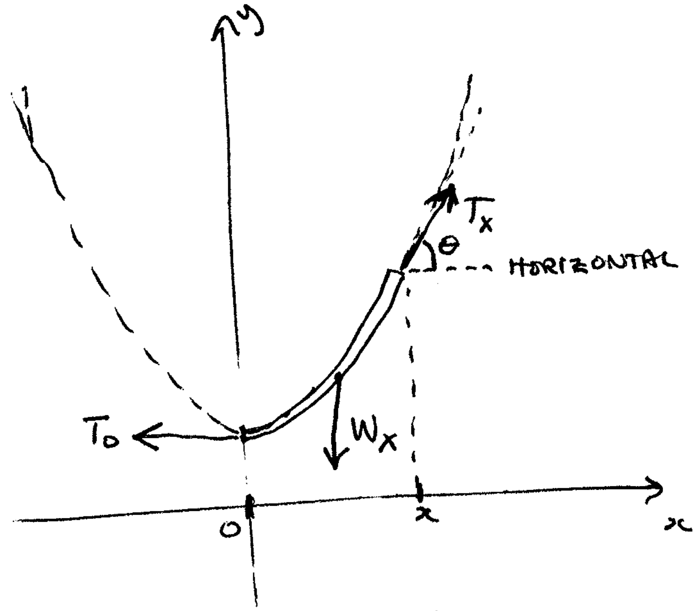
The CABLE EQUATION

Graph of a function
$y = f(x)$
= shape of cable.

Problem: determine equation governing $f(x)$.



Portion of cable over the interval $[0, x]$ is <u>not moving</u>.

$\Rightarrow$ forces acting on it all balance out.

$\Rightarrow$ $T_x \sin\theta = W_x$ (vertical forces balance),

and $T_x \cos\theta = T_0$ (horizontal forces balance).

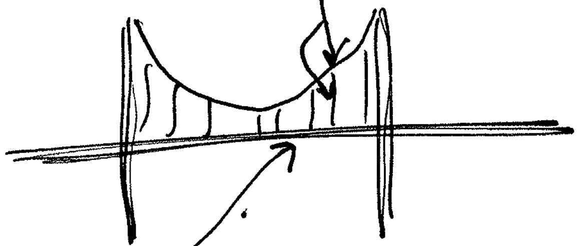
$$\Rightarrow \quad \tan\theta = \frac{W_x}{T_0}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{W_x}{T_0}}$$ The CABLE Eqⁿ (*)

<u>Application/Case</u> ① <u>Suspension bridge</u>

99% of weight, W_x, comes from the horizontal "road" portion of the bridge — ignore the suspension cables!

ignore weight from here and here!



Assume, all weight comes from here.

With this simplification

$$W_x \propto x$$

$$W_x = kx \quad \left(\begin{array}{c} k \\ \text{some} \\ \text{constant} \end{array}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{k}{T_0} x$$

$$\Rightarrow \quad \boxed{y = \frac{k}{2T_0} x^2 + C}$$

a parabola!

Application / Case ② Freely hanging cable.

Uniform cable assumption

$W_x \propto$ length of cable over $[0, x]$.

$W_x = k$ (length of cable over $[0,x]$)

... k = some const.

Cable Eqn ($*$) becomes

$$\frac{dy}{dx} = \frac{k}{T_0}\left(\text{length of cable over } [0,x]\right)$$

see [7] below

$$= \frac{k}{T_0}\left(\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx\right)$$

Let $v = \frac{dy}{dx}$

$$v = \frac{k}{T_0} \int_0^x \sqrt{1+v^2}\, dx$$

$\frac{d}{dx} \Rightarrow$ (using Fund Th$^{\underline{m}}$ of Calc on R.H.S. ....)

$$\frac{dv}{dx} = \frac{k}{T_0} \sqrt{1+v^2}$$

$$\Rightarrow \int_0^v \frac{dv}{\sqrt{1+v^2}} = \int_0^x \frac{k}{T_0}\, dx$$

$$\Rightarrow \sinh^{-1}(v) = \frac{k}{T_0} x$$

$$\Rightarrow v = \sinh\left(\frac{k}{T_0} x\right)$$

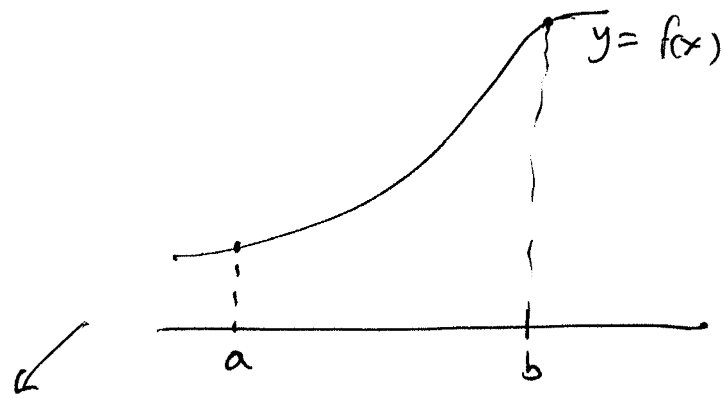
$$\Rightarrow \frac{dy}{dx} = \sinh\left(\frac{k}{T_0} x\right)$$

$$\Rightarrow \boxed{y = \frac{T_0}{k} . \cosh\left(\frac{k}{T_0} x\right) + C}$$

a scaled hyperbolic cosine graph!

$\boxed{\text{Catenary curve}}$

Finally let's address [+] . . . .

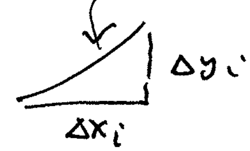


Compute length of portion of $y=f(x)$ graph over $[a,b]$ as follows.

(1) Chop curve into little bits (using a partition of $[a,b]$)

(2) Pythagoras $\Rightarrow$ estimate for length of each little bit

length $\approx \sqrt{\Delta x_i^2 + \Delta y_i^2}$



(3) Sum + take limits

$$\text{Length} = \lim_{\substack{\#\text{ partition} \\ \text{points} \to \infty \\ (\text{size of bits} \to 0)}} \sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \lim \sum_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i$$ . . . A Riemann Sum!!

Limit of Riemann sums $\equiv$ definite integral.

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

expression for arclength!