MATH 2924

Midterm-II
SOLUTIONS

#1 $\int x \sin(2x)\, dx$

using integration by parts

$$\int u\,dv = uv - \int v\,du$$

let: $u = x \Rightarrow du = dx$

$dv = \sin(2x)\,dx \Rightarrow v = \int \sin(2x)\,dx$

$$= -\frac{\cos(2x)}{2}$$

$$\therefore \int x \sin(2x)\,dx = -\frac{x\cos(2x)}{2} - \int \left(-\frac{\cos(2x)}{2}\right) dx$$

$$= -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4}$$

#2 $$\int \frac{dx}{\sqrt{x^2+4x}} = \int \frac{dx}{\sqrt{x^2+4x+4-4}}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2-4}}$$

let $x+2 = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$$\Rightarrow \int \frac{dx}{\sqrt{x^2+4x}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{\sqrt{4\sec^2\theta-4}}$$

$$= \int \frac{2\sec\theta\tan\theta}{2\tan\theta}\,d\theta \quad \left\{ \begin{array}{l} \text{we} \\ \sec^2\theta - 1 = \tan^2\theta \end{array} \right\}$$

$$\therefore \int \frac{dx}{\sqrt{x^2+4x}} = \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$x+2 = 2\sec\theta$

$$\Rightarrow \tan\theta = \sqrt{\sec^2\theta - 1} = \frac{\sqrt{(x+2)^2 - 4}}{2} = \frac{\sqrt{x^2+4x}}{2}$$

$$\therefore \int \frac{dx}{\sqrt{x^2+4x}} = \ln\left|\frac{x+2}{2} + \frac{\sqrt{x^2+4x}}{2}\right| + c$$

(A)

$\boxed{\#3}$ $$\frac{2x^2 - x + 4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4) + x(Bx+c)}{x(x^2+4)}$$

comparing Numerators

$$2x^2 - x + 4 = A(x^2+4) + (Bx+c)x$$

$$= (A+B)x^2 + cx + 4A$$

comparing coefficients $\quad A+B = 2$

$c = -1$

$4A = 4 \Big\} \Rightarrow A = 1$

$A = 1 \Rightarrow 1+B = 2$

$\Rightarrow B = 1$

$\therefore A = 1, B = 1, C = -1$

$$\therefore \quad \frac{2x^2-x+4}{x^3+4x} = \frac{1}{x} + \frac{x-1}{x^2+4}$$

$$\therefore \int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{x}{x^2+4}\,dx \quad \rightsquigarrow \quad \begin{aligned} u &= x^2+4 \\ du &= 2x\,dx \end{aligned}$$

$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+4|$$

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right), \quad \text{use } \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right).$$

$$\therefore \int \frac{2x^2-x+4}{x^3+4x}\,dx = \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

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$\boxed{\#4}$ $$\lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty} \frac{1}{x}\ln x}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \rightarrow \frac{\infty}{\infty} \text{ form}$$

use L'Hopitals Rule

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\therefore \lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty} \frac{\ln x}{x}} = e^0 = \underline{1}$$

$\boxed{\#5}$ $$\int_2^{\infty} \frac{dx}{x\ln x}$$

consider

$$\lim_{b\to\infty} \int_2^{b} \frac{dx}{x\ln x}, \quad \text{let } u = \ln x$$
$$du = \frac{1}{x}dx$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u}du = \lim_{b\to\infty} [\ln|u|]_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \ln|\ln b| - \ln|\ln 2|$$

$$= \infty$$

$$\therefore \int_2^{\infty} \frac{dx}{x\ln x} \text{ is divergent}$$