PRINT NAME: SOLUTIONS

Calculus IV [2443–002] Quiz II

Q1]... State the second derivative test for functions of two variables.

Ans: Let (a, b) satisfy $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Define

$$D(x, y) \;=\; (f_{xx})(f_{yy}) - (f_{xy})^2$$

- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local minimum point.
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local maximum point.
- If $D(a, b) < 0$ and $f_{xx}(a, b) > 0$, then (a, b) is neither a local max nor a local min point [Saddle].
- (If $D(a, b) = 0$, the test is inconclusive.)

Q2]... Find and test the critical points of the function

$$f(x, y) \;=\; xye^{-(x^2+y^2)/2}\,.$$

Soln: We compute the first derivatives using the product and chain rules.

$$f_x = (1 - x^2)ye^{-(x^2+y^2)/2} \qquad\qquad f_y = (1 - y^2)xe^{-(x^2+y^2)/2}\,.$$

Since e to any power is always positive, we see that $f_x = 0 = f_y$ if and only if $(1 - x^2)y = 0 = (1 - y^2)x$, and that these equations are true if and only if (x, y) is one of the following five points: $(0, 0), (1, 1), (1, -1), (-1, 1), (-1, -1)$.

Now, the second derivatives work out to be

$$f_{xx} \;=\; -2xye^{-(x^2+y^2)/2} - xy(1 - x^2)e^{-(x^2+y^2)/2}\,,$$

and

$$f_{yy} \;=\; -2xye^{-(x^2+y^2)/2} - xy(1 - y^2)e^{-(x^2+y^2)/2}\,,$$

and

$$f_{xy} \;=\; (1 - x^2)(1 - y^2)e^{-(x^2+y^2)/2}\,.$$

Thus, we can evaluate the second derivatives at the critical points to get.

- $D(0, 0) \;=\; 0^2 - 1^2 \;=\; -1 \;<\; 0$ implies a saddle point at $(0, 0)$.
- $D(-1, 1) \;=\; D(1, -1) \;=\; (2e^{-1})^2 - 0^2 \;>\; 0$, and $f_{xx}(-1, 1) \;=\; f_{xx}(1, -1) \;=\; 2e^{-1} \;>\; 0$ implies a local minimum at $(-1, 1)$ and at $(1, -1)$.
- $D(-1, -1) \;=\; D(1, 1) \;=\; (-2e^{-1})^2 - 0^2 \;>\; 0$, and $f_{xx}(-1, -1) \;=\; f_{xx}(1, 1) \;=\; -2e^{-1} \;<\; 0$ implies a local maximum at $(-1, -1)$ and at $(1, 1)$.