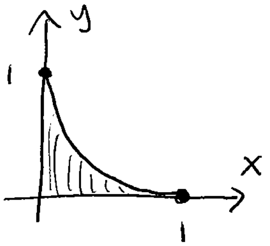
Mid I – Sol'ns

Spring '08

Q1]... [**15 points**] Determine the area below the graph $\sqrt{x}+\sqrt{y}=1$ over the interval $[0,1]$.



$$\text{Area} = \int_0^1 y\,dx \qquad \ldots \text{Now } \sqrt{y} = 1-\sqrt{x} \Rightarrow y = (1-\sqrt{x})^2 = 1+x-2\sqrt{x}$$

$$= \int_0^1 (1+x-2x^{\frac{1}{2}})\,dx$$

$$= \left[x + \frac{x^2}{2} - \frac{4}{3}x^{3/2}\right]_0^1$$

$$= 1+\frac{1}{2}-\frac{4}{3} = \frac{1}{2}-\frac{1}{3} = \boxed{\frac{1}{6}}$$

Now suppose you divide the interval $[0,1]$ into 1,000 equal width subintervals, and use Riemann sums

$$R_{1{,}000} = \sum_{i=1}^{1{,}000} f(x_i^*)\Delta x$$

to approximate the area above. On one hand you take x_i^* to be right endpoints of the subintervals and obtain one Riemann sum $R^r_{1{,}000}$. On the other hand you may take x_i^* to be left endpoints of the subintervals and obtain a second Riemann sum $R^l_{1{,}000}$. Which one of the following statements is true? Give reasons for your answer.

1. $R^l_{1{,}000}$ gives a closer approximation to the area than $R^r_{1{,}000}$.
2. $R^r_{1{,}000}$ gives a closer approximation to the area than $R^l_{1{,}000}$.
3. $R^r_{1{,}000}$ and $R^l_{1{,}000}$ give equally close approximations to the area.

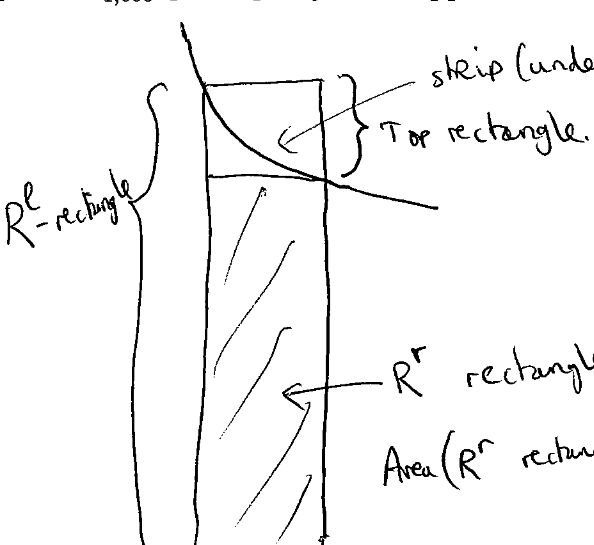
$y = 1+x-2x^{\frac{1}{2}}$

$y' = 1-x^{-\frac{1}{2}} < 0$ on $[0,1]$

$y'' = \frac{1}{2}x^{-3/2} > 0$ on $[0,1]$

$\Downarrow$

Curve is decreasing & concave up (as in the picture!)



Area(R^r rectangle) is closer to strip area than Area(R^l rectangle) since the graph cuts below the diagonal of the top rectangle.

Q2]... **[12 points]** Express the following limit as a definite integral, and then evaluate the integral.

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{2}{n\left(1+\frac{3i}{n}\right)^2}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}\right)\cdot\frac{1}{\left(1+\frac{3i}{n}\right)^2}$$

Think of $x_i^* = \frac{3i}{n}$

starts at $\frac{3}{n} \approx 0$

& ends at $\frac{3n}{n} = 3$

$\boxed{\text{interval} = [0,3]}$

then $\Delta x = \frac{3}{n}$

& $\frac{2}{n} = \frac{2}{3}(\Delta x)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2 \quad \Delta x}{3(1+x_i^*)^2}$$

$$= \int_0^3 \frac{2}{3}\frac{dx}{(1+x)^2} \quad \ldots$$

Let $u = 1+x$

$du = dx$

$x=3 \Rightarrow u=4$

$x=0 \Rightarrow u=1$

you could get this at start by picking $u_i^* = 1+\frac{3i}{n}$

(interval from $1+0=1$ to $1+3=4$ eh...)

$$= \int_1^4 \frac{2}{3} u^{-2}\,du = \frac{2}{3}\frac{u^{-1}}{(-1)}\Bigg]_1^4$$

$$= \frac{2}{3}\left(1-\frac{1}{4}\right) = \frac{2}{3}\left(\frac{3}{4}\right) = \boxed{\frac{1}{2}}$$

Q3]... **[16 points]** Determine the following indefinite integrals.

$$\cancel{\int \frac{\cos^3(x)}{\sin^3(x)}\,dx} \qquad \int \frac{\cos^3(x)}{\sin^5(x)}\,dx$$

$$\int = \int \frac{\cos^2(x)\cos(x)\,dx}{\sin^5(x)} \quad \ldots \text{ write } \cos^2(x) = 1-\sin^2(x)$$

$$= \int \frac{(1-\sin^2(x))}{\sin^5(x)} \cos(x)\,dx \quad \text{--- let } u=\sin(x),\ du = \cos(x)\,dx$$

$$= \int \frac{1-u^2}{u^5}\,du = \int u^{-5} - u^{-3}\,du = \frac{u^{-4}}{-4} - \frac{u^{-2}}{-2} + C$$

$$= \frac{1}{2\sin^2(x)} - \frac{1}{4\sin^4(x)} + C$$

Can also get
$-\frac{1}{4}\cot^4(x) + C$
Check these are same!!

$$\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$$

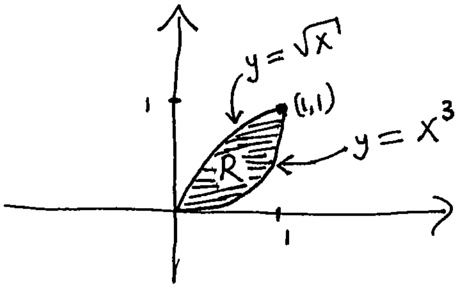
Let $v = 1+\sqrt{x}$ $\quad dv = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx \quad 2\,dv = \frac{dx}{\sqrt{x}}$

$$\int = \int \sqrt{v}\,2\,dv = \int 2\,v^{\frac{1}{2}}\,dv$$

$$= 2\,\frac{v^{3/2}}{3/2} + C$$

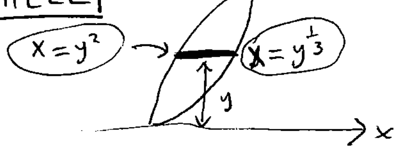
$$= \frac{4}{3}(1+\sqrt{x})^{3/2} + C$$

Q4]... [**20 points**] In each case below, use the cylindrical shell method to write down an integral for the volume of revolution of the given region about the given line. In each case, also use the washer method to write down a second integral for the volume of revolution. You do **not** have to evaluate any of the resulting integrals.



- The region R about the x-axis.

SHELL

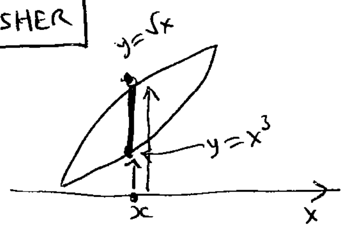


thickness

$$\text{Vol}=\int_0^1 2\pi\,(\text{radius})(\text{length})\,dy$$

$$=\int_0^1 2\pi\cdot y\cdot\left(y^{\frac{1}{3}}-y^2\right)dy$$

WASHER

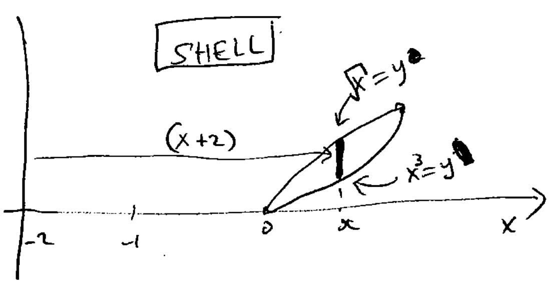


$$\text{Vol}=\int_0^1 \pi\left(r_{out}^2-r_{in}^2\right)dx$$

$$=\int_0^1 \pi\left((\sqrt{x})^2-(x^3)^2\right)dx$$

- The region R about the line $x=-2$.

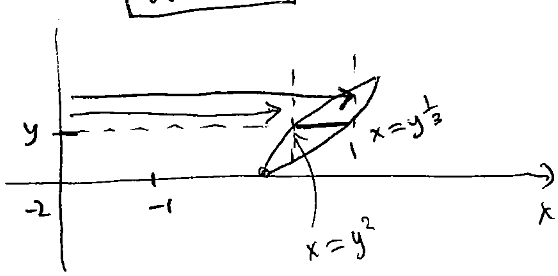
SHELL



$$\text{Vol}=\int_0^1 2\pi\,(\text{radius})(\text{length})\,dx$$

$$=\int_0^1 2\pi\,(x+2)\left(\sqrt{x}-x^3\right)dx$$

WASHER



$$\text{Vol}=\int_0^1 \pi\left((r_{out})^2-(r_{in})^2\right)dy$$

$$=\int_0^1 \pi\left(\left(2+y^{\frac{1}{3}}\right)^2-\left(2+y^2\right)^2\right)dy$$

Q5]... **[12 points]** Show that the following two functions are indeed equal.

$$\int_0^x \left(\int_0^u f(t)\, dt \right) du = \int_0^x (x-u) f(u)\, du$$

[Hint: One way to show that $f(x) = g(x)$ is to start off by showing that $f'(x) = g'(x)$...]

Using the Hint...

$$\frac{d}{dx}(\text{LHS}) = \frac{d}{dx}\left(\int_0^x \left(\int_0^u f(t)\, dt \right) du \right) = \int_0^x f(t)\, dt \quad \left(\text{by Fund. Th}^{\text{m}}\right)$$

$$\frac{d}{dx}(\text{RHS}) = \frac{d}{dx}\left(\int_0^x (x-u) f(u)\, du \right) = \frac{d}{dx}\left(x \int_0^x f(u)\, du - \int_0^x u f(u)\, du \right)$$

sum & product rules $\longrightarrow$ $$= \frac{dx}{dx} \cdot \int_0^x f(u)\, du + x \cdot \frac{d}{dx}\left(\int_0^x f(u)\, du \right) - \frac{d}{dx}\left(\int_0^x u f(u)\, du \right)$$

$$= \int_0^x f(u)\, du + \underbrace{\cancel{x \cdot f(x)}}_{\text{Fund Th}^{\text{m}}} - \underbrace{\cancel{x f(x)}}_{\text{Fund Th}^{\text{m}}}$$

$$= \int_0^x f(u)\, du.$$

So $$\frac{d}{dx}(\text{LHS} - \text{RHS}) = \int_0^x f(t)\, dt - \int_0^x f(u)\, du = 0$$

$\Rightarrow$ LHS $-$ RHS $= C$ (a constant).

When $x = 0$ we get: LHS $= \int_0^0 (\ldots) = 0 =$ RHS

$\Rightarrow$ $0 - 0 = C$ $\Rightarrow$ $C = 0$

$\Rightarrow$ LHS $-$ RHS $= 0$ i.e. $\boxed{\text{LHS} = \text{RHS}}$ done!