Honors Problem Set II — Applications of Integration

Q1. Buffon's Needle Problem.

(a) Do Q11 on page 601 of the textbook.

(b) Can you think of a way of using part (a) to estimate the value of π?

(c) Implement your idea in part (b)! What estimate did you get for π?

Q2. Geometric interpretation of parameter in hyperbolic functions. Do Q68 on page 470 of the textbook.

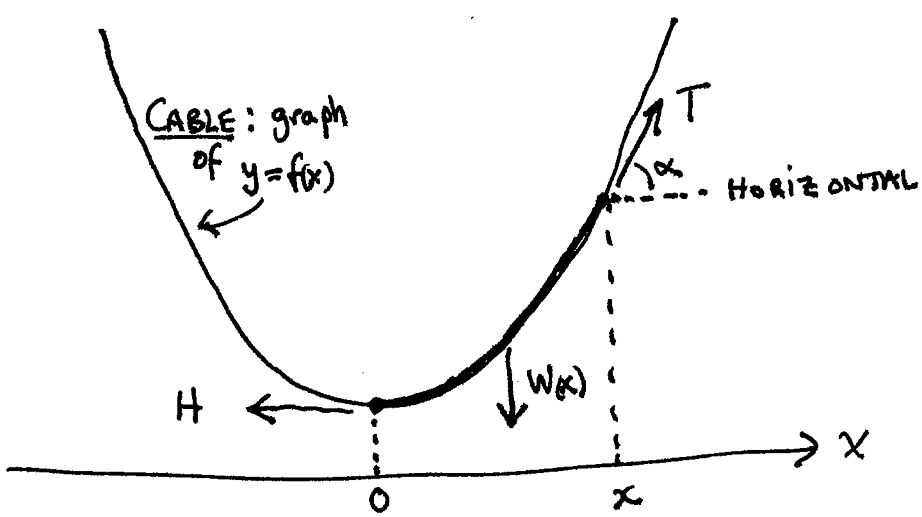
Q3. The Cable Equation. We want to determine the shape of a cable (eg. a cable in a suspension bridge, or a freely hanging chain). The basic idea is to think of the shape of the cable as being the graph of a function $y = f(x)$.

Our goal is to determine an equation (the cable equation) which the function $y = f(x)$ must satisfy, given the physics of the cable. We then examine two special cases of this general cable equation: one for a cable in a suspension bridge, and one for a freely hanging cable.

[PHYSICS $\longrightarrow$ MATH]
Focus on the portion of the cable which lies over the interval $[0, x]$ on the x-axis. There are 3 forces acting on this portion of cable:

1. tension T at the top portion of the cable (parallel to the cable).
2. tension H at the lowest point of the cable (horizointal).
3. weight $W(x)$ of the portion of the cable.

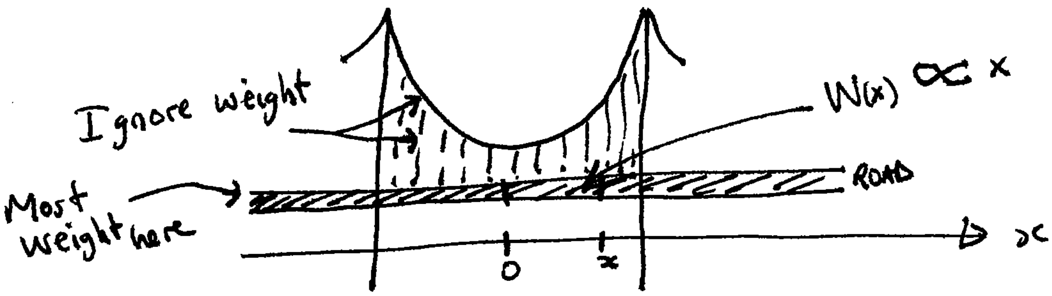
Since the cable is not moving, the physics gives two equations.
Cable is not moving horizontally $\longrightarrow T\cos(\alpha) = H$.
Cable is not moving vertically $\longrightarrow T\sin(\alpha) = W(x)$.
Q3(a). Show how these equations follow from the physics statements.



Q3(b). Combine the previous two equations, and remember some Calc I, to get the Cable Equation:

$$\frac{dy}{dx} = \frac{W(x)}{H}$$

Suspension Bridge Example. Ignore the weight of the suspension cables. All weight comes from the horizontal part of the bridge (which contains the asphalt etc). This is a uniform road. Thus $W(x)$ is proportional to length x along the road. This gives an equation $W(x) = kx$ for some constant k which may depend on the density of the road and the acceleration due to gravity.
Q3(c). Solve the Cable Eqn in this case. What shape do you get?



Freely Hanging Cable Example – The Catenary. In this case $W(x)$ is proportional to the length of the portion of the cable which lies over the interval $[0, x]$. We shall see later in this course that an integral gives us arc-length. Thus we get

$$W(x) = k\int_0^x \sqrt{(dt)^2 + (dy)^2} = k\int_0^x \sqrt{1 + (dy/dt)^2}\, dt$$

Q3(d). Let $v = dy/dx$. Show that the Cable Eqn becomes

$$\frac{dv}{dx} = \frac{k}{H}\sqrt{1+v^2}$$

Q3(e). Divide across by $\sqrt{1+v^2}$. Now take $\int dx$ of both sides. Show that you eventually get

$$\sinh^{-1}(v) = \frac{k}{H}x + C$$

and say why the constant C must be zero.
Q3(f). Finally, remember that $v = dy/dx$, and solve for y to get

$$y = \frac{H}{k}\cosh\left(\frac{k}{H}x\right) + D$$

What is the physical significance of the constant D?