Honors Problem Set III

Integration by parts, polynomial approximations of functions, and irrationality of e.

Overview. The first part of this homework set asks you to use integration by parts (many times!) to show that a "reasonable" function $f(x)$ can be approximated by a polynomial in an interval about the input point 0. We express the error (difference of the polynomial and the function values) as a definite integral. The coefficients of the approximating polynomials can be expressed in terms of high derivatives of f at 0.

Next you are to explore some of these approximating polynomials. Draw some graphs and list some output values for the functions e^x, $\sin(x)$ and $\cos(x)$.

Finally, we use the value of the polynomial for e^x at $x = 1$ together with the integral error term, to give a slick proof that e is not a rational number.

Integration by parts and approximating polynomials. Let $f(x)$ be a function which has derivatives of all orders. Our starting point is one of the key ideas in this course, the Fundamental Theorem of Calculus:

$$f(x) - f(0) = \int_0^x f'(t)\,dt$$

1. Rewrite this as
$$f(x) = f(0) + \int_0^x f'(t)\,dt$$
and we see that it says that $f(x)$ is approximated by the constant function $f(0)$ with error given by $\int_0^x f'(t)\,dt$. This is not terribly exciting.

2. Do integration by parts on the integral term with $u = f'(t)$ and $dv = dt$. Just be a little weird when it comes to writing down v. Note that $v = t$ up to a constant, choose the constant to be the negative of the upper limit x, and write
$$v = t - x$$
Do the integration by parts (remember t is the variable, and x is a constant) and see that you indeed get
$$f(x) = f(0) + f'(0)x - \int_0^x (t - x)f''(t)\,dt$$

3. Rewrite this as
$$f(x) = f(0) + f'(0)x + \int_0^x (x - t)f''(t)\,dt$$
and note that it says that $f(x)$ is approximated by the straight line function $y = f(0) + f'(0)x$ with error equal to $\int_0^x (x - t)f''(t)\,dt$.

4. What is a common name for the straight line $y = f(0) + f'(0)x$?

5. Now do integration by parts on the integral term $\int_0^x (x-t)f''(t)\,dt$ in the previous expression. Take $u = f''(t)$ and $dv = (x-t)dt$. Check that you indeed get

$$f(x) \;=\; f(0) \,+\, f'(0)x \,+\, \frac{f''(0)}{2}x^2 \,+\, \int_0^x \frac{(x-t)^2}{2} f^{(3)}(t)\,dt$$

This says that the function $f(x)$ is approximated by the polynomial $f(0) + f'(0)x + \frac{f''(0)}{2}x^2$ with error term given by the integral $\int_0^x \frac{(x-t)^2}{2} f^{(3)}(t)\,dt$.

6. Do two more steps of the definite integration and write out the corresponding polynomial approximations for $f(x)$.

7. In general, after n steps, you get the following expression

$$f(x) \;=\; f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

The last expression above is called "the Taylor polynomial approximation for $f(x)$ on an interval about 0 with an integral form of the remainder (error)". You'll have lots of fun with this in Calculus III. In particular you'll think about what happens as $n \to \infty$, and will investigate objects called "Taylor series". We'll denote the polynomial by $T_n(x)$ in honor of Taylor. Thus the last equation becomes

$$f(x) \;=\; T_n(x) + \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

Examples of functions and their approximating polynomials. In this section we investigate some functions $f(x)$ and their corresponding $T_n(x)$ polynomials. We also see how to give an upper bound on the error in the e^x example.

1. Write down $T_1, T_2, \ldots, T_9$ for the function $f(x) = \sin(x)$. What patterns do you notice?

   Using a graphing utility (eg. *Grapher* for the mac) plot $y = \sin(x)$ and $T_n(x)$ on the same graph. Do a separate graph for $n = 3$, $n = 5$, $n = 7$ and $n = 9$.

2. Write down $T_1, T_2, \ldots, T_8$ for the function $f(x) = \cos(x)$. What patterns do you notice?

   Using a graphing utility (eg. *Grapher* for the mac) plot $y = \cos(x)$ and $T_n(x)$ on the same graph. Do a separate graph for $n = 2$, $n = 4$, $n = 6$ and $n = 8$.

3. Write down $T_1, T_2, \ldots, T_6$ for the function $f(x) = e^x$. Using a graphing utility (eg. *Grapher* for the mac) plot $y = e^x$ and $T_n(x)$ on the same graph. Do a separate graph for $n = 3$, $n = 4$, $n = 5$ and $n = 6$. Evaluate $T_n(1)$ for $n = 2, 3, 4, 5, 6$ and compare your answers with e^1. Verify that $e^1 - T_n(1)$ is never 0 and is strictly smaller than $\frac{1}{n!}$ in these cases.

4. Now show that the inequality above is always the case (not just for $n = 2, \ldots, 6$). Do this by noticing that $f^{(n+1)}(t) = e^t$ is less than or equal to the constant function $y = e$ on the interval $[0, 1]$. Thus, for $x = 1$, the integral error term is no larger than

$$\int_0^1 \frac{(x-t)^n}{n!} e\,dt$$

   Compute this integral, and check that it is positive and always strictly smaller than $\frac{1}{n!}$ for $n \geq 2$.

The proof that e is not a rational number. You are now psychologically prepared (even better, mathematically prepared!) to see that e is not a rational number. Drum roll...

It all depend on the following fact which we established in the previous section using approximating polynomials and integral error terms.

"For each integer $n \geq 2$, the number e can be approximated by the finite sum

$$1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

with a positive error ϵ_n which is strictly smaller than $\frac{1}{n!}$."

1. Suppose that e were a rational number. That is, $e = p/q$ for some pair of integers p and q. Note that $q \geq 2$ (why?).

2. Now take $n = q$ and write

$$e - \left(1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right) = \epsilon_q$$

   where $0 < \epsilon_q < \frac{1}{q!}$.

3. Multiply both sides of this equation by $q!$. The left side of the resulting equation is an integer (why?).

4. The right side of the resulting equation gives a contradiction (why?).

5. We conclude that e is not rational (why?).