SOLUTIONS — CALC II — SP'08

Q1]... **[10 points]** Evaluate the following limit by first converting it into a suitable form for l'Hospital's rule.

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{x\ln(x) - 1(x-1)}{(x-1)\ln(x)}\right)$$

← this is in $\frac{0}{0}$ form!

$$\underset{\text{L'Hôpital's rule (0/0 form)}}{=} \lim_{x\to 1^+}\left(\frac{1\ln(x) + \frac{x}{x} - 1}{1\ln(x) + \left(\frac{x-1}{x}\right)}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{\ln(x)}{\ln(x) + \frac{x-1}{x}}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{x\ln(x)}{x\ln(x) + (x-1)}\right)$$

$$\underset{\text{L'Hôpital's rule (0/0 form)}}{=} \lim_{x\to 1^+}\left(\frac{1\ln(x) + \frac{x}{x}}{1\cdot\ln(x) + \frac{x}{x} + 1}\right) = \frac{0+1}{0+1+1}$$

$$= \boxed{\frac{1}{2}}$$

Q2]... [10 points] Evaluate the indefinite integral

$$\int \sin^4(x)\, dx$$

$$= \int \left(\sin^2(x)\right)^2 dx$$

$$= \int \left(\frac{1 - \cos(2x)}{2}\right)^2 dx$$

$$= \frac{1}{4} \int 1 + \cos^2(2x) - 2\cos(2x)\, dx$$

$$= \frac{1}{4} \int 1 + \frac{1 + \cos(4x)}{2} - 2\cos(2x)\, dx$$

$$= \frac{1}{4}\left[\frac{3x}{2} + \frac{\sin(4x)}{8} - \sin(2x)\right] + C$$

Q3]... **[10 points]** Evaluate the following improper integral

$$\int_e^\infty \frac{\ln(x)dx}{x^{10}}$$

$$= \lim_{T\to\infty} \left(\int_e^T \frac{\ln(x)}{x^{10}}\, dx \right)$$

Int by parts

$u = \ln(x) \to du = \frac{dx}{x}$

$dv = x^{-10}dx \to v = -\frac{1}{9}x^{-9}$

$$= \lim_{T\to\infty} \left[\frac{-\ln(x)}{9x^9}\Big|_e^T + \frac{1}{9}\int_e^T \frac{dx}{x^{10}} \right]$$

$$= \lim_{T\to\infty} \left(\frac{1}{9e^9} - \frac{\ln(T)}{9T^9} - \frac{1}{81}\frac{1}{x^9}\Big|_e^T \right)$$

$$= \lim_{T\to\infty} \left(\frac{1}{9e^9} - \underbrace{\cancel{\frac{\ln(T)}{9T^9}}}_{\to 0} + \frac{1}{81e^9} - \underbrace{\cancel{\frac{1}{81T^9}}}_{\to 0} \right)$$

$$= \frac{1}{9e^9} + \frac{1}{81e^9}$$

Q4]... **[10 points]** Find an expression for the inverse of the following function

$$y = \frac{10^x - 10^{-x}}{2}$$

① $x \leftrightarrow y$:

$$x = \frac{10^y - 10^{-y}}{2}$$

② Solve for "new y" :

$$2x = 10^y - \frac{1}{10^y}$$

$$(10^y)^2 - 2x(10^y) - 1 = 0$$

Quadratic $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\rightarrow$

$$10^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$10^y = \frac{2x + \sqrt{4x^2 + 4}}{2}$$

$$10^y = x + \sqrt{x^2+1}$$

Take $+\sqrt{}$ Since $10^y > 0$

$$\ln(10^y) = \ln\left(\sqrt{x^2+1} + x\right)$$

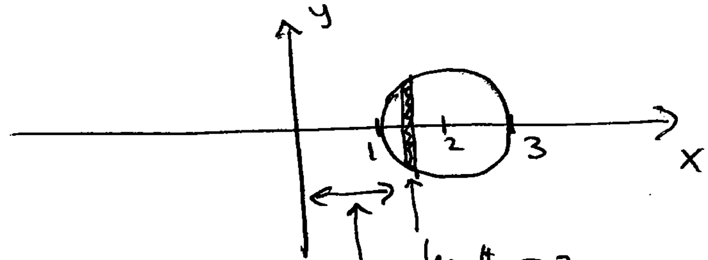
$= y \ln(10)$

$$y = \frac{\ln\left(\sqrt{x^2+1} + x\right)}{\ln(10)}$$

Q5]... **[10 points]** Using the SHELL method, determine the volume (of the "bagel") obtained by rotating the region inside the circle

$$(x-2)^2 + y^2 = 1$$

about the y-axis.



length $= 2y$
radius $= x$
thickness $= dx$

$$\text{Vol} = \int 2\pi\,(\text{radius})\,(\text{length})\,(\text{thickness})$$

$$= \int_1^3 2\pi\, x\, (2y)\, dx$$

$$= \int_1^3 2\pi\, x\, 2\sqrt{1-(x-2)^2}\, dx$$

$$= 4\pi \int_1^3 x\sqrt{1-(x-2)^2}\, dx$$

Trig Subst$^{\underline{n}}$:

Let $(x-2) = \sin\theta$

$dx = \cos\theta\, d\theta$

$\sqrt{} = \cos\theta$

limits are $\pm \pi/2$

$$= 4\pi \int_{-\pi/2}^{\pi/2} (2+\sin\theta)\cos\theta\, \cos\theta\, d\theta$$

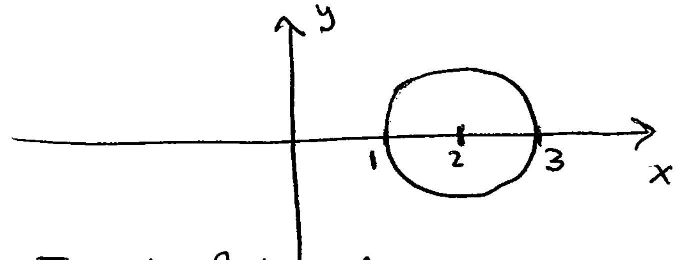
$$= 8\pi \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta + \cancel{4\pi \int_{-\pi/2}^{\pi/2} \sin\theta\, \cos^2\theta\, d\theta} = 0 \quad \text{(odd integrand!)}$$

$$= 8\pi \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos(2\theta)}{2}\right) d\theta = 4\pi \left[\theta + \cancel{\frac{\sin(2\theta)}{2}}_{\,0}\right]_{-\pi/2}^{\pi/2} = \boxed{4\pi^2}$$

Q6]... [**15 points**] Determine the surface area (of the "bagel") obtained by rotating the circle

$$(x-2)^2 + y^2 = 1$$

about the y-axis.



Surface Area = 2 Surface Area of Top $\frac{1}{2}$ of bagel.

$$= 2 \int 2\pi \,(\text{radius})\, ds$$

$$= 2 \int 2\pi \; x \; ds$$

$$= 4\pi \int_1^3 x \sqrt{1+(y')^2}\, dx$$

$$(x-2)^2 + y^2 = 1$$
$$\downarrow$$
$$2(x-2) + 2yy' = 0$$
$$\downarrow$$
$$y' = -\left(\frac{x-2}{y}\right)$$
$$\sqrt{1+(y')^2} = \sqrt{\frac{y^2+(x-2)^2}{y^2}}$$
$$= \frac{1}{y}$$

$$= 4\pi \int_1^3 \frac{x}{y}\, dx$$

$$= 4\pi \int_1^3 \frac{x \; dx}{\sqrt{1-(x-2)^2}}$$

As before
$(x-2) = \sin\theta$
$dx = \cos\theta \, d\theta$
$\sqrt{\;} = \cos\theta$
Limits $= \pm \pi/2$

Trig Subst

$$= 4\pi \int_{-\pi/2}^{\pi/2} \frac{(2+\sin\theta)}{\cancel{\cos\theta}} \cancel{\cos\theta}\, d\theta$$

$$= 4\pi \int_{-\pi/2}^{\pi/2} 2 \, d\theta + \cancel{4\pi \int_{-\pi/2}^{\pi/2} \sin\theta \, d\theta} \to 0$$

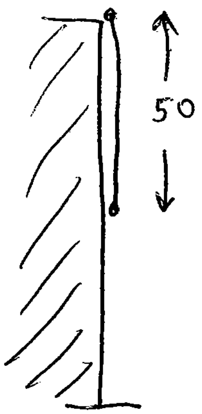
odd integrand

$$= \boxed{8\pi^2}$$

Q7]... [10 points] Write down the work done when the constant force F lbs moves an object through a distance d ft.

$$\text{Work} = (\text{force})(\text{distance}) = F \cdot d \quad \text{ft-lbs}$$

A uniform 50 ft long rope weighing 2 lb/ft hangs over the edge of a tall building as shown. Using Riemann sums, show how to obtain an integral expression for the work done in lifting the rope to the top of the building. Give reasons for your steps.



Chop rope into n equal length pieces, each of length $\Delta x = \frac{50}{n}$

ith piece has weight $= 2\Delta x = 2\Delta x_i$

ith piece will travel height of x_i^* (between x_{i-1} & x_i) to top of building

$\Rightarrow$ Work done $= 2x_i^* \Delta x_i$

$\Rightarrow$ Total work done for all n pieces $= \sum_{i=1}^{n} 2x_i^* \Delta x_i$

Now take limit as $n \to \infty$ to get continuous version ...

$$\text{Work} = \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^* \Delta x_i = \int_0^{50} 2x \, dx$$

What is the work done in lifting the rope above to the top of the building? That is, evaluate the integral you found above.

$$\text{Work} = \int_0^{50} 2x \, dx = \cancel{2}\frac{x^2}{\cancel{2}}\Big]_0^{50} = 2500 \text{ ft-lbs.}$$

Q8]...[15 points] Find the values of A, B and C which make the following (partial fractions) equation true.

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\frac{1}{(x-1)(x^2+1)} = \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)}$$

Comparing Numerators gives

$$\begin{cases} 0 = A + B & \text{--- (coeff of } x^2) \\ 0 = C - B & \text{--- (coeff of } x^1) \\ 1 = A - C & \text{--- (coeff of } x^0) \end{cases}$$

$-A = B = C$

$2A = 1$

$A = \frac{1}{2}$ $\quad$ $B = C = -\frac{1}{2}$

Now, evaluate the integral

$$\int \frac{dx}{(x-1)(x^2+1)}$$

$$\int = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{\cancel{x}+1}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{2x\,dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{1+x^2}$$

$u = x^2+1$ substn

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\tan^{-1}(x) + C$$

Q9]... [**10 points**] Determine if the function

$$F(x) = \int_0^{2x-x^2} \cos\left(\frac{1}{1+t^2}\right) dt$$

has any maximum or minimum values?

$F(x)$ is differentiable (by Fund Th$^{\underline{m}}$), and

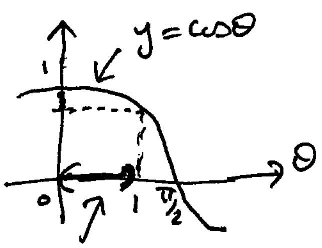
$$F'(x) = \cos\left(\frac{1}{1+(2x-x^2)^2}\right)\cdot(2-2x)$$

Fund Th^{m} + Chain Rule.

<u>Note</u> ... $(2x-x^2)^2$ ranges between 0 & ∞

$\Rightarrow \dfrac{1}{1+(2x-x^2)^2}$ ranges between $\dfrac{1}{1+0}$ & $\dfrac{1}{1+\infty}$

ie. 1 & 0.



$0 \le \theta \le 1 < \pi/2$

$\Rightarrow \cos\left(\dfrac{1}{1+(2x-x^2)^2}\right)$ is always $+$.

So ... $F'(x) = 0$ only when $2-2x = 0$

ie. $\boxed{x=1}$

& sign of $F'(x)$ = sign of $(2-2x)$

$-$ for $x > 1$

$+$ for $x < 1$

$\Rightarrow$ $F(x)$ has local max at $x=1$

& No other critical points.