SP 2011 — Math 3333-003 — Mid III

Q1]... [20 points] Say whether each of the following statements is True or False.

(1) If A is a $p \times q$ matrix, then $\text{rank}(A) + \text{nullity}(A) = p$.

False $\leftarrow$ if $p \neq q$ ... $\text{rank}(A) + \text{nullity}(A) = q$

(2) The $n \times n$ matrix A is non-singular if and only if $\text{nullity}(A) = 0$.

TRUE

$$\begin{aligned} A \text{ nonsingular} &\iff \text{Kernel}(A) = \{0\} \\ &\iff \dim(\text{Kernel}(A)) = 0 \\ &\iff \text{nullity}(A) = 0 \end{aligned}$$

(3) Any two linearly independent collections of vectors in a vector space V have the same number of vectors.

False eg: $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ are both l.i. in $\mathbb{R}^3$.

(4) Any two bases for a vector space V have the same number of vectors.

TRUE Theorem from class notes.

(5) The dimension of the vector space of polynomials of degree at most n is $n+1$.

TRUE basis = $\{1, x, x^2, \dots, x^n\}$ has $n+1$ elements.

Q2]... **[20 points]** Give the definition of the *dimension* of a vector space.

If V has basis $\{\vec{v}_1, -, \vec{v}_n\}$, then $\dim(V) = n$.

If V does not have a finite basis, then $\dim(V) = \infty$.

Find a basis for the vector space spanned by the vectors $\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}11\\10\\7\end{bmatrix}, \begin{bmatrix}7\\6\\4\end{bmatrix}$. What is the dimension of this space?

$$\begin{pmatrix}\vec{v}_1^T\\ \vec{v}_2^T\\ \vec{v}_3^T\\ \vec{v}_4^T\end{pmatrix} = \begin{pmatrix}1&2&2\\3&2&1\\11&10&7\\7&6&4\end{pmatrix} \underset{\text{equivalent}}{\overset{\text{row}}{\sim}} \begin{pmatrix}1&2&2\\0&-4&-5\\0&-12&-15\\0&-8&-10\end{pmatrix}$$

$$\sim \begin{pmatrix}1&2&2\\0&4&5\\0&0&0\\0&0&0\end{pmatrix}$$

$$\Rightarrow \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \operatorname{Span}\left\{\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}0\\4\\5\end{pmatrix}\right\}$$

& these are l.i.

has dimension 2

$$\left(\text{basis} = \left\{\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}0\\4\\5\end{pmatrix}\right\}\right)$$

Q3]... [20 points] a) Give the definition of the *rank* of a matrix.

$$\text{rank}(A) = \begin{cases} \dim(\text{row space}(A)) = \text{row rank}(A) \\ \underline{\text{OR}} \text{ (equivalently)} \\ \dim(\text{col space}(A)) = \text{col rank}(A) \end{cases}$$

b) Prove that $\text{rank}(AB) \le \text{rank}(A)$ for all matrices A, B for which the product AB is defined.

$$\boxed{\begin{array}{l} \text{rank}(A) = \text{col rank}(A) \\ \text{rank}(AB) = \text{col rank}(AB) \end{array}}$$

Let A be $m \times n$ & B be $n \times p$

$$\begin{aligned} j\text{th column of } AB &= A(j\text{th column of } B) \\ &= A\begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} \\ &= b_{1j}\,\text{col}_1(A) + \cdots + b_{nj}\,\text{col}_n(A) \\ &= \text{linear combination of columns of } (A). \end{aligned}$$

Since every column of AB is a l.c. of columns of A, then

$$\text{col space}(AB) \subseteq \text{col space}(A) \Rightarrow \dim(\text{col space}(AB)) \le \dim(\text{col space}(A))$$
$$\Rightarrow \text{rank}(AB) \le \text{rank}(A)$$

(c) Give an example which shows that $\text{rank}(AB)$ may be strictly smaller than $\text{rank}(A)$.

eg $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \text{rank}(A) = 2$

$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Then $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which has rank $= 0 \lneq 2$

Another way to think of the proof that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.

$A_{m \times n}$ $\qquad$ $B_{n \times p}$

$$\mathbb{R}^p \xrightarrow{B} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

$\mathrm{rank}(A) = \dim(\mathrm{colspace}(A)) = \dim(\mathrm{range}(A)) = \dim(A(\mathbb{R}^n))$ —— ①

$\mathrm{rank}(AB) = \dim(\mathrm{colspace}(AB)) = \dim(\mathrm{range}(AB)) = \dim(AB(\mathbb{R}^p))$ —— ②

But $B(\mathbb{R}^p) \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$

$\Rightarrow$ $A(B(\mathbb{R}^p)) \subseteq A(\mathbb{R}^n)$

i.e. $AB(\mathbb{R}^p) \subseteq A(\mathbb{R}^n)$ is a subspace

$\Rightarrow$ $\dim(AB(\mathbb{R}^p)) \leq \dim(A(\mathbb{R}^n))$

Now use ① & ② to conclude

$$\mathrm{rank}(AB) \leq \mathrm{rank}(A)$$

It's the same proof as the previous one $\left(\begin{array}{l} \text{rank} = \text{col rank} \\ \quad = \dim(\text{col space}) \\ \quad = \dim(\text{range}) \\ \quad \text{etc} \ldots \end{array}\right)$.

Another proof of $\boxed{\operatorname{rank}(AB) \le \operatorname{rank}(A)}$

$A_{m\times n}$ $\quad$ $B_{n\times p}$

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "B"] \arrow[rr, bend right, "AB"'] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathbb{R}^p & \mathbb{R}^n \arrow[l, "B^T"'] & \mathbb{R}^m \arrow[l, "A^T"'] \arrow[ll, bend left, "(AB)^T"] \end{tikzcd}$$

$$\operatorname{rank}(A^T) + \operatorname{nullity}(A^T) = m \qquad — (1)$$

$$\operatorname{rank}((AB)^T) + \operatorname{nullity}((AB)^T) = m \qquad — (2)$$

But $(AB)^T = B^T A^T$

so if $\vec{v} \in \mathbb{R}^m$ is in $\operatorname{Null}(A^T)$

$\Rightarrow A^T \vec{v} = \vec{0}$

$\Rightarrow B^T A^T \vec{v} = B^T \vec{0} = \vec{0}$

& so ~~B^TA^T~~ $\vec{v} \in \operatorname{Null}(B^T A^T) = \operatorname{Null}((AB)^T)$.

Thus, $\operatorname{Null}(A^T) \underset{\substack{\uparrow \\ \text{subspace}}}{\subseteq} \operatorname{Null}((AB)^T)$

$\Rightarrow \dim(\operatorname{Null}(A^T)) \le \dim(\operatorname{Null}(AB)^T)$

$\Rightarrow \operatorname{Nullity}(A^T) \le \operatorname{Nullity}((AB)^T)$

(1) & (2) $\Rightarrow$ $m - \operatorname{rank}(A^T) \le m - \operatorname{rank}((AB)^T)$

$\Rightarrow \underbrace{\operatorname{rank}((AB)^T)}_{\operatorname{rank}(AB)} \le \operatorname{rank}(A^T) = \operatorname{rank}(A)$

done!

Q4]. . . [20 points] Let

$$S = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$$

be an ordered basis for $\mathbb{R}^3$.

Find the change of coordinates matrix C from the S basis to the standard basis for $\mathbb{R}^3$.

$$C = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 0\\1 & 3 & 1\end{bmatrix}_{\text{Stand} \leftarrow S}$$

Let $\mathbf{v} = \begin{bmatrix}-1\\4\\5\end{bmatrix}$. Find the coordinates of $\mathbf{v}$ with respect to the basis S.

If $\vec{v} = \begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}_S$, then $\begin{bmatrix}-1\\4\\5\end{bmatrix}_{\text{Stand}} = C_{\text{Stand} \leftarrow S} \begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}_S$.

Therefore the coordinates $\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix}$ of $\vec{v}$ w.r.t. basis S are the solution to the linear system:

$$\begin{pmatrix}1 & 1 & 1\\1 & 2 & 0\\1 & 3 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} = \begin{pmatrix}-1\\4\\5\end{pmatrix}$$

<u>Gaussian Elimination</u>

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & -1\\1 & 2 & 0 & 4\\1 & 3 & 1 & 5\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 1 & -1\\0 & 1 & -1 & 5\\0 & 2 & 0 & 6\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 1 & -1\\0 & 1 & -1 & 5\\0 & 0 & 2 & -4\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1 & 1 & 1 & -1\\0 & 1 & -1 & 5\\0 & 0 & 1 & -2\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 0 & 1\\0 & 1 & 0 & 3\\0 & 0 & 1 & -2\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & -2\\0 & 1 & 0 & 3\\0 & 0 & 1 & -2\end{array}\right]$$

Actually we did full Gauss-Jordan elimination!

Coordinates of $\vec{v}$ w.r.t. S are $\begin{bmatrix}-2\\3\\-2\end{bmatrix}_S$.

Q5]... [20 points] The matrix A below defines a linear map $\mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{v} \mapsto A\mathbf{v}$.

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Find the matrix of this linear map with respect to the ordered basis

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

in the domain, and the ordered basis

$$T = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

in the range.

$$\begin{tikzcd}
\mathbb{R}^2_{\text{, stand.}} \arrow[r, "\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}"] & \mathbb{R}^2_{\text{, stand.}} \\
\mathbb{R}^2_{,S} \arrow[u, "\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}"] \arrow[r, "B"] & \mathbb{R}^2_{,T} \arrow[u, swap, "\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}"]
\end{tikzcd}$$

Matrix of the linear map $\vec{v} \longmapsto A\vec{v}$, w.r.t. basis S in domain & basis T in codomain is

$$B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} -4 & -3 \\ 5 & 3 \end{pmatrix} = \boxed{\begin{pmatrix} -4/3 & -1 \\ 5/3 & 1 \end{pmatrix}}$$

Ans

You should check that $A \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\frac{4}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{5}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

and $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.