MATH 5863-001 Topology II Spring 2011

Free Groups and Free Products

1. Prove that if $g, h \in F(X) - \{1\}$ and $[g, h] = 1$, then $\langle g, h \rangle \cong \mathbb{Z}$.

   Hint: use normal forms, and induction on $|g| + |h|$. Note that one can always arrange that g is cyclically reduced without increasing $|g| + |h|$.

2. Prove that if $g \in F(X) - \{1\}$, then the centralizer $C(g) \cong \mathbb{Z}$.

3. Let $\langle\langle A \rangle\rangle$ denote the normal subgroup generated by A. Prove that $A * B/\langle\langle A \rangle\rangle$ is isomorphic to B.

4. If $g \in A * B$ has cyclically reduced normal form of length greater than 1, then g has infinite order.

5. Prove that if $A \neq 1$ and $B \neq 1$, then the center $Z(A * B) = 1$.

6. Prove that if $g \in A * B$ and $ord(g) < \infty$, then g is conjugate to an element of A or to an element of B.

7. Let $m(G)$ denote the maximum of the orders of finite order elements of the group G. Prove that $m(A * B) = \max\{m(A), m(B)\}$.

8. Let $[G, G]$ denote the commutator subgroup of the group G. Prove that if $G = \mathbb{Z}_m * \mathbb{Z}_n$, then $G/[G, G] \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

9. If $\mathbb{Z}_m * \mathbb{Z}_n \cong \mathbb{Z}_p * \mathbb{Z}_q$, then $\{m, n\} = \{p, q\}$. Hint, use problems 7 and 8.

10. Let $m, n \in \mathbb{Z}^+$ and let $G_{m,n} = \langle a, b \mid a^m = b^n \rangle$.

    (a) Prove that $\langle a^m \rangle < Z(G_{m,n})$, and so $\langle a^m \rangle \lhd G_{m,n}$.

    (b) Prove that $G_{m,n}/\langle a^m \rangle \cong \mathbb{Z}_m * \mathbb{Z}_n$.

    (c) Use problem 5 above to conclude that $\langle a^m \rangle = Z(G_{m,n})$. Thus (b) becomes

    $$\frac{G_{m,n}}{Z(G_{m,n})} \cong \mathbb{Z}_m * \mathbb{Z}_n .$$

    (d) Now prove that if $G_{m,n} \cong G_{p,q}$, then $\{m, n\} = \{p, q\}$.