Math 5863-001 Final Examination Name:________________________

Friday, May 9, 2003, 8:00am–10:00am.

Answer as many questions as you can (stomach).

Q1].. Write down a Wirtinger presentation for $\pi_1(S^3 \setminus \text{Fig-8 knot})$.

Determine (showing your work) the abelianization of $\pi_1(S^3 \setminus \text{Fig-8 knot})$.

Prove that there does not exist a retraction from the space X (which is defined to be S^3 minus an open tubular neighborhood of the Fig-8 knot) to the boundary torus of X.

Q2].. Use covering spaces (as in the proof of the Kurosh theorem) to give a complete description of the kernel of the homomorphism

$$\psi : \langle a \,|\, a^2 \rangle * \langle b \,|\, b^3 \rangle \;\to\; \mathrm{Perm}(\{1,2,3\}) \;\; : a \mapsto (12), \;: b \mapsto (123)$$

Q3].. Here are two infinite regular covering spaces of the bouquet of two circles. Determine the automorphism group in each case. [Say how the automorphisms act on the covering spaces, and say why you have listed all of the automorphisms in each case]

Q4].. State the Path Homotopy Lifting Property (PHLP) and the Path Lifting Property (PLP) for covering spaces $p : \tilde{X} \to X$.

Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a path-connected covering space. Let $G = \pi_1(X, x_0)$ and $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$. Prove (sketch) that the map

$$\Phi : p^{-1}(x_0) \to G/H \ : \tilde{x_1} \mapsto H[p \circ \gamma]$$

where γ is a path in $\tilde{X}$ from $\tilde{x_0}$ to $\tilde{x_1}$, is well-defined, injective, and surjective. Conclude that the cardinality of $p^{-1}(x_0)$ equals the index of H in G.

Well-defined

Injective [This requires a little thought]

Surjective

Q5].. By constructing a covering space of the bouquet of two circles, give a detailed description of the subgroup H of the free group $F_{\{a,b\}}$ which is generated as follows:

$$\langle\, aba,\ baa,\ abba\bar{b},\ b\bar{a}\bar{b}a \,\rangle$$

- Determine the index of H in $F_{\{a,b\}}$.
- Is H free (if so describe a free basis for H)?
- Is H normal in $F_{\{a,b\}}$?
- Check whether $a^3b^3 \in H$.