SP 2014 ; Mid II ; Solutions

Q1]...[20 points] Find the derivatives of the following functions. In each case, show the steps of your work (giving the names of differentiation rules that you use).

$$y = \sqrt{x}\cos(x)$$

PRODUCT RULE

$$\frac{dy}{dx} = \frac{d(\sqrt{x})}{dx} \cdot \cos(x) + \sqrt{x} \cdot \frac{d}{dx}\left(\cos(x)\right)$$

$$= \frac{1}{2\sqrt{x}}\cos(x) + \sqrt{x}\left(-\sin(x)\right)$$

Power Rule

Trig. diff.

$$= \frac{\cos(x)}{2\sqrt{x}} - \sqrt{x}\sin(x)$$

$$y = \frac{2\sin(x)}{x^3 - 2x + 4}$$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{\frac{d\,2\sin(x)}{dx}(x^3-2x+4) - (2\sin x)\frac{d}{dx}(x^3-2x+4)}{(x^3-2x+4)^2}$$

Trig. diff.
Poly. diff.

$$= \frac{2\cos(x).(x^3-2x+4) - 2\sin(x).(3x^2-2)}{(x^3-2x+4)^2}$$

Q2]... **[20 points]** Find the derivative of the following function. Show the steps of your work, and name the differentiation rules that you use.

$$y = \sqrt{\sin(x^2+4)}$$

CHAIN RULE

$$\frac{dy}{dx} = \frac{d\sqrt{u}}{du} \cdot \frac{du}{dx} \qquad u = \sin(x^2+4)$$

$$= \frac{d\sqrt{u}}{du} \cdot \frac{d\sin v}{dv} \cdot \frac{dv}{dx} \quad \text{... ch. Rule again}$$

$$v = x^2+4$$

$$= \frac{1}{2\sqrt{u}} \cdot \cos(v) \cdot (2x+0)$$

Power Rule; Trig. diff.; Poly. diff.

$$= \frac{2x\cos(x^2+4)}{2\sqrt{\sin(x^2+4)}}$$

Find the derivative y' of the function y defined implicitly by the equation

$$x^2 + xy = y^3$$

$$\frac{d}{dx}(x^2 + xy) = \frac{d}{dx}(y^3)$$

$$2x + \underbrace{\frac{dx}{dx}y + x\frac{dy}{dx}}_{\text{Product Rule}} = 3y^2 \cdot \frac{dy}{dx}$$

ch. Rule (implicit).

$$2x + y = \frac{dy}{dx}(3y^2 - x)$$

$$\frac{dy}{dx} = \frac{2x+y}{3y^2-x}$$

Q3]...[20 points] Write down the value of the following limit

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

Use the limit above to compute the value of the following limit (showing your work)

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$$

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{1-\cos(x)}{x^2}\left(\frac{1+\cos(x)}{1+\cos(x)}\right)$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos(x))}$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos(x)}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot\lim_{x\to 0}\left(\frac{1}{1+\cos x}\right) = 1\cdot 1\cdot\frac{1}{1+1}$$

$$= \boxed{\frac{1}{2}}$$

If $f(x) = \cos(3x)$ find an expression for the 87th derivative, $f^{(87)}(x)$. Show your work.

Remainder 0 $\quad f^{(0)}(x) = f(x) = \cos(3x)$

" 1 $\quad f'(x) = -3^1\sin(3x)$

" 2 $\quad f^{(2)}(x) = -3^2\cos(3x)$

" 3 $\quad f^{(3)}(x) = +3^3\sin(3x)$

$4\overline{)87}$ 21 R3

$\Rightarrow +\sin(3x)$

& "power of 3" = 87

$$\boxed{f^{(87)}(x) = 3^{87}\sin(3x)}$$

↑

- Trig functions cycle in 4's
- Powers of 3 come from chain Rule term $\frac{d(3x)}{dx}$ at each stage.

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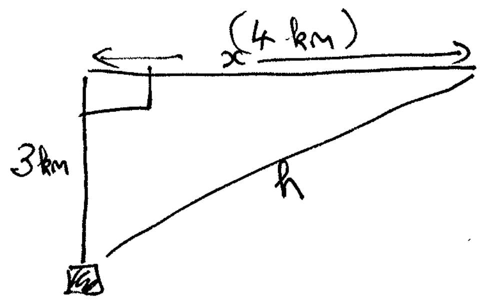
Q4]... [20 points] An aircraft is flying horizontally at a rate of 4 km/minute at an altitude of 3 km. The aircraft passes directly overhead a radar station at 3pm. How fast is the distance between the aircraft and the radar station increasing (one minute later) at 3:01pm?



Let x = horizontal distance from Radar station flyover point at time t

Let h = actual distance from radar station at time t

Pythag $\Rightarrow$ $3^2 + x^2 = h^2$

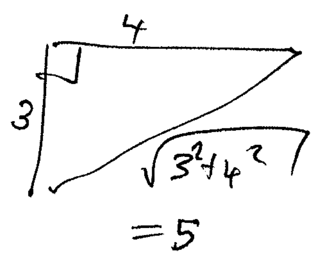
$$\Rightarrow \frac{d}{dt}\left(3^2 + x^2\right) = \frac{d}{dt}\left(h^2\right)$$

Ch. Rule $\Rightarrow$ $0 + \cancel{2}x\frac{dx}{dt} = \cancel{2}h\frac{dh}{dt}$

$$\boxed{\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt}}$$

We're told $\frac{dx}{dt} = 4\text{km/min}$

At 3:01 pm, $x = (4).(1) = 4\text{ km}$



$\Rightarrow h = 5\text{ km}$.

$$\Rightarrow \frac{dh}{dt} = \frac{4}{5}(4) = \frac{16}{5}\text{ km/min.}$$

Q5]...[20 points] Write down the expression for the linearization of the differentiable function $f(x)$ at the input a.

$$L(x) = f(a) + f'(a)(x-a)$$

Find the linearization of the function $y = \sqrt{x}$ at the input 25. Show your work.

$$f(x) = y = \sqrt{x} \qquad y' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f(25) = \sqrt{25} = 5 \qquad f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{(2)(5)} = \frac{1}{10}$$

$$L(x) = 5 + \frac{1}{10}(x - 25)$$

Use the linearization above to find an approximation for $\sqrt{26}$. Show your work.

$$\begin{aligned}\sqrt{26} &= f(26) \\ &\approx L(26) \\ &= 5 + \frac{1}{10}(26-25) \\ &= 5 + \frac{1}{10}(1) \\ &= 5.1\end{aligned}$$