Vector fields in 3-dimensions and Grad, Curl, and Div

$$\left\{\begin{array}{l}\text{Functions}\\ f(x,y,z)\\ \text{on a}\\ \text{domain}\\ E \text{ in } \mathbb{R}^3.\end{array}\right\} \xrightarrow{\text{grad}} \left\{\begin{array}{l}\text{Vector fields}\\ \mathbf{F}=\langle P,Q,R\rangle\\ \text{on the domain}\\ E \text{ in } \mathbb{R}^3.\end{array}\right\} \xrightarrow{\text{curl}} \left\{\begin{array}{l}\text{Vector fields}\\ \mathbf{F}=\langle P,Q,R\rangle\\ \text{on the domain}\\ E \text{ in } \mathbb{R}^3.\end{array}\right\} \xrightarrow{\text{div}} \left\{\begin{array}{l}\text{Functions}\\ f(x,y,z)\\ \text{on the}\\ \text{domain}\\ E \text{ in } \mathbb{R}^3.\end{array}\right\}$$

1. It is best to think about grad, curl, and div in 3-dimensions in terms of a single vector differential operator (called "Del" or "Nabla")

$$\nabla \;=\; \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

   This differential operator can take functions and return vector fields (grad), and can also take vector fields and return either vector fields (curl) or functions (div).

2. The first operator, *grad*, takes a function $f = f(x,y,z)$ and returns the vector field

$$\text{grad}(f) \;=\; \nabla f \;=\; \langle f_x, f_y, f_z \rangle\,.$$

3. The second operator, *curl*, takes a vector field $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ and returns another vector field

$$\text{curl}(\mathbf{F}) \;=\; \nabla \times \mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \;=\; \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

4. The third operator, *div*, takes a vector field $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ and returns a function

$$\text{div}(\mathbf{F}) \;=\; \nabla \cdot \mathbf{F} \;=\; P_x + Q_y + R_z$$

5. It is easy to check that the composition of two successive operators is zero.

   (a) The first pair is

$$\text{curl} \circ \text{grad} \;=\; 0\,.$$

   That is,

$$\nabla \times (\nabla f) \;=\; \nabla \times \langle f_x, f_y, f_z \rangle \;=\; \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle \;=\; \langle 0, 0, 0 \rangle$$

   for all functions f.

   (b) The second pair is

$$\text{div} \circ \text{curl} \;=\; 0\,.$$

   That is,

$$\nabla \cdot (\nabla \times \mathbf{F}) \;=\; \nabla \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \;=\; R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \;=\; 0$$

   for all vector fields $\mathbf{F} = \langle P, Q, R \rangle$.

6. **Tests to see if a vector field has a scalar or vector potential.**

   (a) Suppose the vector field $\mathbf{F}$ is equal to ∇f for some function f (we say that $\mathbf{F}$ is conservative, and that it has a scalar potential). Then $\nabla \times \mathbf{F} = \nabla \times (\nabla f) = \mathbf{0}$.

   In particular, if $\mathbf{F}$ is a vector field for which $\nabla \times \mathbf{F} \neq \mathbf{0}$, then you can conclude that $\mathbf{F}$ is NOT the gradient of some function f.

   (b) Suppose the vector field $\mathbf{F}$ is equal to $\nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$ (we say that $\mathbf{F}$ has a vector potential). Then $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{G}) = 0$.

   In particular, if $\mathbf{F}$ is a vector field for which $\nabla \cdot \mathbf{F} \neq 0$, then you can conclude that $\mathbf{F}$ is NOT the curl of some vector field $\mathbf{G}$.

7. **Two Questions.** This gives rise to two questions.

   (a) Suppose the vector field $\mathbf{F}$ satisfies $\nabla \times \mathbf{F} = \mathbf{0}$. Is it always the case that $\mathbf{F}$ is the gradient of some function f?

   (b) Suppose the vector field $\mathbf{F}$ satisfies $\nabla \cdot \mathbf{F} = 0$. Is it always the case that $\mathbf{F}$ is the curl of some vector field $\mathbf{G}$?

   The answers to these questions will be similar to the 2-dimensional setting. *It depends on the connectivity properties of the region E. If the region E has particular types of "holes" the answers will be "No," otherwise "Yes."*

   We will investigate these questions further after we have learned about Stokes' Theorem and the Divergence Theorem.

8. **The Laplacian.** The Laplacian is denoted by ∇^2 (some people use the symbol Δ for the Laplacian) and is defined to be div $\circ$ grad. It takes functions and returns functions (which involve second derivatives of the input function).

$$\Delta f \;=\; \nabla^2 f \;=\; \nabla \cdot \nabla f \;=\; \nabla \cdot \langle f_x, f_y, f_z \rangle \;=\; f_{xx} + f_{yy} + f_{zz}$$

9. **Two interpretations of Green's Theorem.** Let C be a closed curve satisfying the hypotheses of Green's Theorem, and bounding a region D. Let $\mathbf{F} = \langle P(x,y), Q(x,y) \rangle$ be a vector field satisfying the hypotheses of Green's Theorem. We shall consider P and Q as functions of three variables (x, y, z) (their values are independent of the third variable z) and shall consider $\mathbf{F}$ to be a vector field in 3-dimensions as follows, $\mathbf{F} = \langle P, Q, 0 \rangle$. Then we have the following.

   (a) **Tangential Version of Green's Theorem.**

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds \;=\; \oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_D \mathrm{curl}(\mathbf{F}) \cdot \hat{\mathbf{k}}\, dA$$

   (b) **Normal Version of Green's Theorem.**

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{N}}\, ds \;=\; \oint_C P\, dy - Q\, dx \;=\; \iint_D P_x + Q_y\, dA \;=\; \iint_D \mathrm{div}(\mathbf{F})\, dA$$