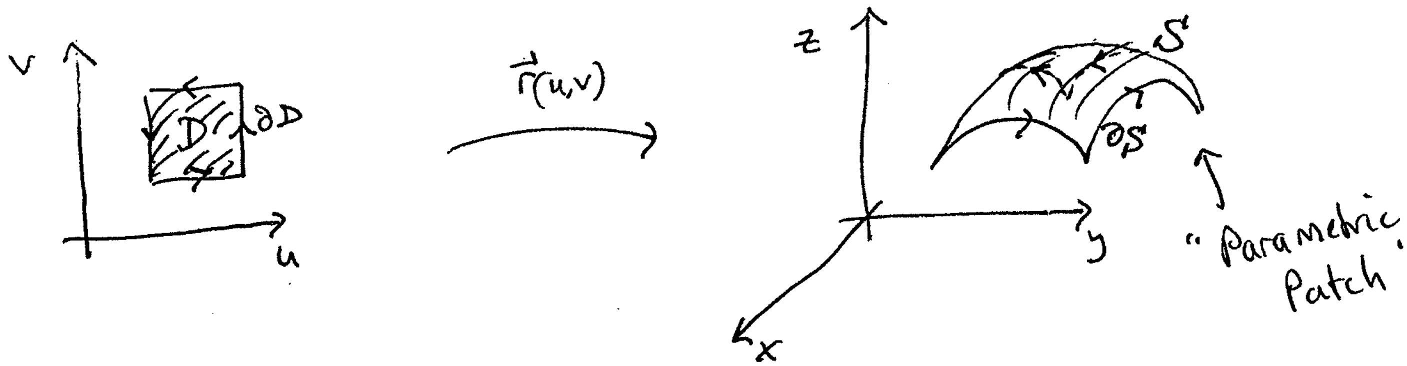
— STOKES' PROOF SKETCH —



$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \oint_{\partial S} P\,dx + Q\,dy + R\,dz$$

$$= \oint_{\partial D} P(x_u\,du + x_v\,dv) + Q(y_u\,du + y_v\,dv) + R(z_u\,du + z_v\,du)$$

$$= \oint_{\partial D} (Px_u + Qy_u + Rz_u)\,du + (Px_v + Qy_v + Rz_v)\,dv$$

$$\overset{\text{Green}}{=} \iint_D (Px_v + Qy_v + Rz_v)_u - (Px_u + Qy_u + Rz_u)_v \,du\,dv$$

$$= \iint_D \begin{matrix} P_u x_v + Q_u y_v + R_u z_v + \cancel{P x_{vu}} + \cancel{Q y_{vu}} + \cancel{R z_{vu}} \\ - P_v x_u - Q_v y_u - R_v z_u - \cancel{P x_{uv}} - \cancel{Q y_{uv}} - R z_{uv} \end{matrix} \,du\,dv$$

$$= \iint_D \left(P_u x_v + Q_u y_v + R_u z_v - P_v x_u - Q_v y_u - R_v z_u\right) du\,dv$$

——— (*)

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \iint_D \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du\,dv$$

$$= \iint_D \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \cdot \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du\,dv$$

$$= \iint_D \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \cdot \langle y_u z_v - y_v z_u, x_v z_u - x_u z_v, x_u y_v - x_v y_u \rangle du\,dv$$

$$= \iint_D (R_y - Q_z)(y_u z_v - y_v z_u) + (P_z - R_x)(x_v z_u - x_u z_v) + (Q_x - P_y)(x_u y_v - x_v y_u)\, du\,dv$$

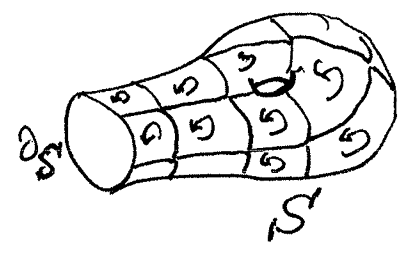
$$= \iint_D (P_x x_u + P_y y_u + P_z z_u) x_v - \cancel{P_x x_u x_v}$$
$$- (P_x x_v + P_y y_v + P_z z_v) x_u + \cancel{P_x x_v x_u}$$
$$du\,dv$$
$$+ \text{ similarly for } Q, R \ \ldots$$

$$= \iint_D (P_u x_v - P_v x_u) + \text{ similarly for } Q, R \quad du\,dv$$

$$= (*)$$

done!

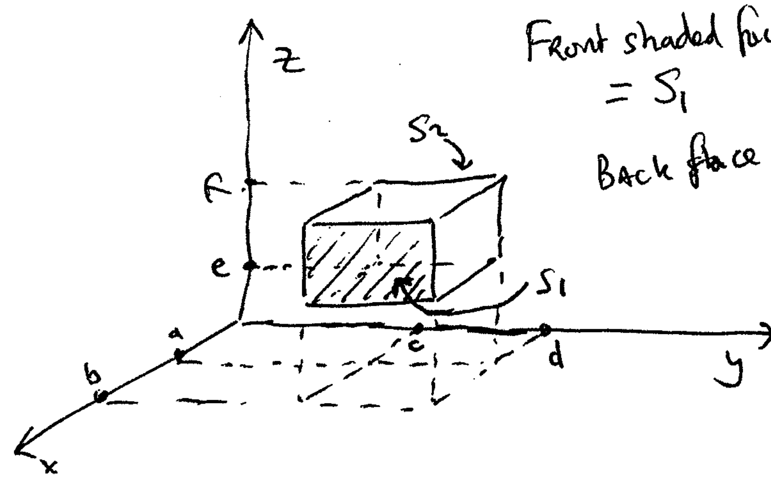
Finally ... chop a general surface into little parametric patches + sum to get general form of Stokes' Th$^{\underline{m}}$.
(inner edge path integrals all cancel)



— DIVERGENCE PROOF SKETCH —

E = Rectangular block

$= [a,b] \times [c,d] \times [e,f]$



Front shaded face $= S_1$

Back face $= S_2$

$$\iiint_E \operatorname{div}(\vec{F})\, dV = \iiint_E (P_x + Q_y + R_z)\, dV$$

$$= \underbrace{\iint_{[c,d]\times[e,f]} \left(\int_a^b P_x \, dx \right) dA}_{} + \iint_{[a,b]\times[e,f]} \left(\int_c^d Q_y \, dy \right) dA + \iint_{[a,b]\times[c,d]} \left(\int_e^f R_z \, dz \right) dA$$

We will just do this term in detail ... other two are similar.

$$1^{st} \text{ term} = \iint_{[c,d]\times[e,f]} P(b,y,z) - P(a,y,z)\, dA$$

$$= \iint_{[c,d]\times[e,f]} \vec{F}(b,y,z) \cdot \underbrace{\hat{\imath}\, dA}_{\substack{\| \\ d\vec{S} \\ (\text{on } S_1)}} + \iint_{[c,d]\times[e,f]} \vec{F}(a,y,z) \cdot \underbrace{(-\hat{\imath})\, dA}_{\substack{\| \\ d\vec{S} \\ (\text{on } S_2)}}$$

$$= \iint_{S_1} \vec{F}(b,y,z) \cdot d\vec{S} + \iint_{S_2} \vec{F}(a,y,z) \cdot d\vec{S}$$

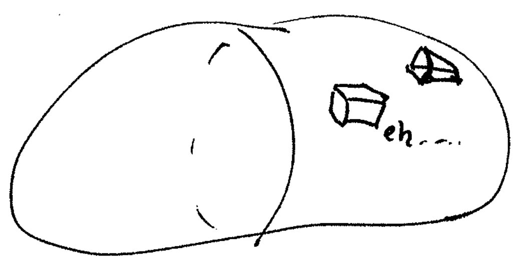
= two (of six) terms of the surface (flux) integral $\iint_{\partial E} \vec{F} \cdot d\vec{S}$.

So ... $\boxed{\iiint_E \operatorname{div}(\vec{F})\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}}$

holds for a rectangular block region.

More generally ... it holds for a region that is simultaneously type I, type II, & type III. —— (**)

Finally, ... chop a general ("potato") E into little regions ("fries") which are of type (**)



Note that all surface (flux) integrals on interior faces cancel out ... left with flux integrals across patches which make up ∂E. — done

$$\boxed{\iiint_E \operatorname{div}(F)\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}}$$