Additional problem 1.

Let D be a subset of \mathbb{R}^2 defined in polar coordinates as

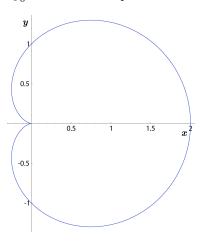
$$D = \{(r, \theta) : \theta \in [a, b], \ q(\theta) < r < f(\theta)\},$$

where a and b are constants with $0 \le a < b \le 2\pi$, and $g : [a, b] \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are functions satisfying $g(\theta) \le f(\theta)$ for all $\theta \in [a, b]$ (see Figure 6 on page 691 for a picture of such a domain).

(a) Use double integral in polar coordinates to show that the area of D is equal to

$$\frac{1}{2} \int_a^b \left[f(\theta)^2 - g(\theta)^2 \right] d\theta .$$

(b) Use the formula from part (a) to find the area inside the curve with equation in polar coordinates is $r=1+\cos\theta$; this curve is called a *cardioid* and is drawn in the figure below. (You may find useful the fact that $\int_0^{\pi/2} \cos^2\theta \, \mathrm{d}\theta = \frac{\pi}{4}$.)



Additional problem 2.

Let a and b be constants with a < b, and let $f : [a,b] \to [0,\infty)$ be a function taking only non-negative values. Consider the curve y = f(z) for $z \in [a,b]$ in the (y,z)-plane in \mathbb{R}^3 (i.e., the points with Cartesian coordinates (0, f(z), z), where $z \in [a,b]$). Rotate this curve around the z-axis to obtain a solid of revolution (see page 357) defined by

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 : z \in [a, b], \sqrt{x^2 + y^2} \le f(z) \right\}.$$

(a) Use triple integral in cylindrical coordinates to show that the volume of E is equal to

$$\int_a^b \pi f(z)^2 dz .$$

(b) Use the formula derived in part (a) to compute the volume of a ball of radius R.