MATH 2934 – Additional problem assigned on 7/8/14

Additional problem.

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$. In class we showed in two different ways that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}(t)| = \frac{\mathbf{r}'(t)\cdot\mathbf{r}(t)}{|\mathbf{r}(t)|} \tag{1}$$

(this is Exercise **13.2**/53 from the book). In this problem you will derive several other expressions for derivatives related to vector functions.

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{|\mathbf{r}(t)|} = -\frac{1}{|\mathbf{r}(t)|^2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}(t)| = -\frac{\mathbf{r}'(t)\cdot\mathbf{r}(t)}{|\mathbf{r}(t)|^3}\ .$$

Hint: Use (1) and the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\phi(t)} = -\frac{1}{\phi(t)^2}\frac{\mathrm{d}}{\mathrm{d}t}\phi(t)$$

for a function ϕ of one variable like in Calculus I (the identity for ϕ follows directly from the Chain Rule for a function of one variable).

(b) Use your result from part (a) to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{r}(t)}{|\mathbf{r}(t)|} = -\frac{\mathbf{r}'(t)\cdot\mathbf{r}(t)}{|\mathbf{r}(t)|^3}\,\mathbf{r}(t) + \frac{\mathbf{r}'(t)}{|\mathbf{r}(t)|}\ .$$

(c) The vector $\mathbf{u}(t) := \dfrac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$ is a unit vector in the direction of $\mathbf{r}(t)$. We know from Example 4 in Section 13.2 that if a vector $\mathbf{u}(t)$ has constant length, then the vector is perpendicular to its derivative, i.e., $\mathbf{u}(t)\cdot\mathbf{u}'(t) = 0$. Use your result from part (b) to show by a direct calculation that

$$\frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{r}(t)}{|\mathbf{r}(t)|} = 0\ ,$$

as it should be.