

**Additional problem 1.**

Let  $D$  be a subset of  $\mathbb{R}^2$  defined in polar coordinates as

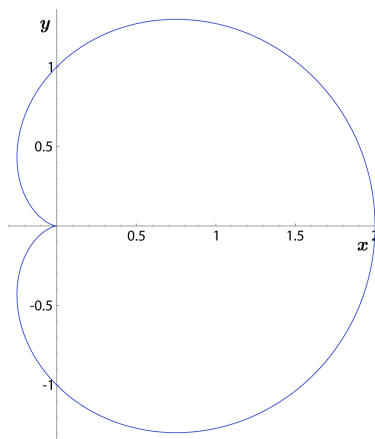
$$D = \{(r, \theta) : \theta \in [a, b], g(\theta) \leq r \leq f(\theta)\} ,$$

where  $a$  and  $b$  are constants with  $0 \leq a < b \leq 2\pi$ , and  $g : [a, b] \rightarrow \mathbb{R}$  and  $f : [a, b] \rightarrow \mathbb{R}$  are functions satisfying  $g(\theta) \leq f(\theta)$  for all  $\theta \in [a, b]$  (see Figure 6 on page 691 for a picture of such a domain).

- (a) Use double integral in polar coordinates to show that the area of  $D$  is equal to

$$\frac{1}{2} \int_a^b [f(\theta)^2 - g(\theta)^2] d\theta .$$

- (b) Use the formula from part (a) to find the area inside the curve with equation in polar coordinates is  $r = 1 + \cos \theta$ ; this curve is called a *cardioid* and is drawn in the figure below. (You may find useful the fact that  $\int_0^{\pi/2} \cos^2 \theta d\theta = \frac{\pi}{4}$ .)

**Additional problem 2.**

Let  $a$  and  $b$  be constants with  $a < b$ , and let  $f : [a, b] \rightarrow [0, \infty)$  be a function taking only non-negative values. Consider the curve  $y = f(z)$  for  $z \in [a, b]$  in the  $(y, z)$ -plane in  $\mathbb{R}^3$  (i.e., the points with Cartesian coordinates  $(0, f(z), z)$ , where  $z \in [a, b]$ ). Rotate this curve around the  $z$ -axis to obtain a solid of revolution (see page 357) defined by

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 : z \in [a, b], \sqrt{x^2 + y^2} \leq f(z) \right\} .$$

- (a) Use triple integral in cylindrical coordinates to show that the volume of  $E$  is equal to

$$\int_a^b \pi f(z)^2 dz .$$

- (b) Use the formula derived in part (a) to compute the volume of a ball of radius  $R$ .