Problem 1.

Let D be a subset of $\mathbb{R}^2$ defined in polar coordinates as

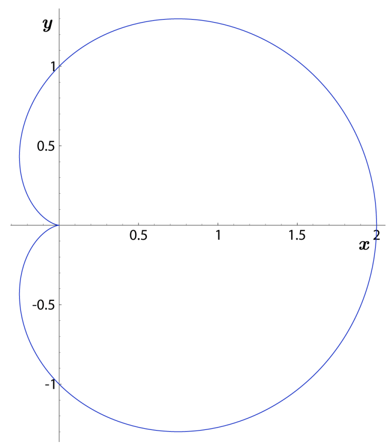
$$D = \{(r,\theta) : \theta \in [a,b]\,,\ g(\theta) \le r \le f(\theta)\}\ ,$$

where a and b are constants with $0 \le a < b \le 2\pi$, and $g : [a,b] \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are functions satisfying $g(\theta) \le f(\theta)$ for all $\theta \in [a,b]$ (see Figure 6 on page 691 for a picture of such a domain).

(a) Use double integral in polar coordinates to show that the area of D is equal to

$$\frac{1}{2}\int_a^b \left[f(\theta)^2 - g(\theta)^2\right] \mathrm{d}\theta\ .$$

(b) Use the formula from part (a) to find the area inside the curve with equation in polar coordinates is $r = 1 + \cos\theta$; this curve is called a *cardioid* and is drawn in the figure below. (You may find useful the fact that $\int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta = \frac{\pi}{4}$.)



Problem 2.

Let a and b be constants with $a < b$, and let $f : [a,b] \to [0,\infty)$ be a function taking only non-negative values. Consider the curve $y = f(z)$ for $z \in [a,b]$ in the (y,z)-plane in $\mathbb{R}^3$ (i.e., the points with Cartesian coordinates $(0, f(z), z)$, where $z \in [a,b]$). Rotate this curve around the z-axis to obtain a solid of revolution (see page 357) defined by

$$E = \left\{(x,y,z) \in \mathbb{R}^3 : z \in [a,b]\,,\ \sqrt{x^2+y^2} \le f(z)\right\}\ .$$

(a) Use triple integral in cylindrical coordinates to show that the volume of E is equal to

$$\int_a^b \pi f(z)^2\,\mathrm{d}z\ .$$

(b) Use the formula derived in part (a) to compute the volume of a ball of radius R.