MATH 2934 – Solution of Problem 15.7/33

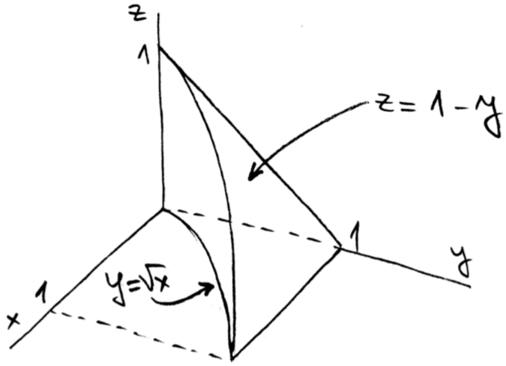

Figure 1: Region of integration E.

1) Writing the integral in the form $I_1 = \int\int\int f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

Method A (from the book): If we consider the 3-dimensional region of integration E as a region of type I, we can write the integral as

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_{D_1}\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,\mathrm{d}z\right]\mathrm{d}A\ ,$$

where the region of integration D_1 in the (x,y)-plane is shown in Figure 2. The functions

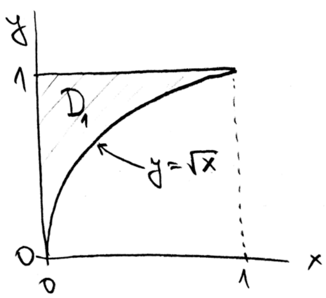

Figure 2: Region of integration in the (x,y)-plane.

giving the limits of integration with respect to z in the inside integral are $u_1(x,y) = 0$ and $u_2(x,y) = 1 - y$ (which means that for a given $(x,y) \in D_1$, the z-coordinate of the points from E that project to this $(x,y) \in D_1$ takes values between 0 and $1 - y$).

Now let us consider the double integral over D_1. If we want to write this integral in the desired order of integration, we have to consider the 2-dimensional region D_1 as a region of type I, i.e., as

$$D_1 = \{(x,y) \in \mathbb{R}^2 | x \in [0,1],\ y \in [\sqrt{x}, 1]\}\ .$$

Taking all this into account, we can write

$$I_1 = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x \ .$$

Method B (alternative): We can also write the integral in the desired form by thinking as follows. First we have to determine the limits of integration of the "outermost" integration, i.e., the integration over x. Clearly, the integration over x goes from 0 to 1, so

$$I_1 = \int_0^1 \int \int f(x, y, z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x \ ,$$

where the other limits of integration are to be determined. For a fixed value of $x \in [0, 1]$, the cross-section of the solid E in Figure 1 with the plane $\{x = (\text{the chosen value } x^*)\}$ looks as shown in Figure 3: in Figure 3(a) it is shown in three dimensions, and in Figure 3(b) you see the projection onto the (y, z)-plane. (Check: When $x^* = 1$, the shaded triangle in

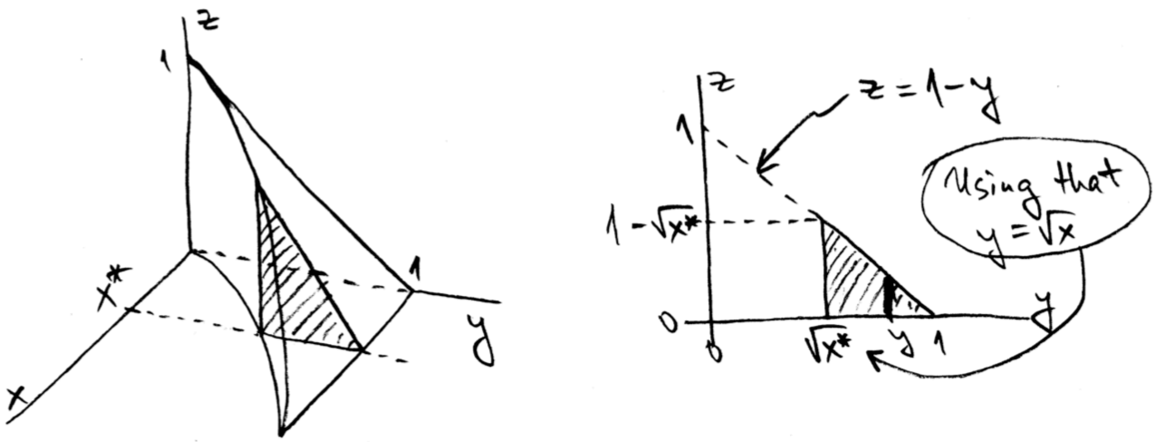

Figure 3: (a) The plane $\{x = x^*\}$ in 3-dim; (b) projection onto the (y, z)-plane.

Figure 3(b) disappears; when $x^* = 0$, we obtain the "big" triangle in the two-dimensional picture.) So the second integration (over y) goes from $\sqrt{x^*}$ to 1:

$$I_1 = \int_0^1 \int_{\sqrt{x}}^1 \int f(x, y, z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x \ .$$

In writing this integral, I omitted the star on x, which was used only temporarily to make things clearer; from now on I will not use such temporary notations. Finally, for a fixed value of y, the variable z is allowed to vary between 0 and $(1 - y)$ (as you can see from segment of the thick line in the shaded region in Figure 3(b)). The result is

$$I_1 = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x \ .$$

2) Writing the integral in the form $I_2 = \int\int\int f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$**.**

Method A (from the book): If we consider the 3-dimensional region of integration E as a region of type III, we can write the integral as

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_{D_3}\left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,\mathrm{d}y\right]\mathrm{d}A\ ,$$

where the region of integration D_3 in the (x,z)-plane is shown in Figure 4. In this case

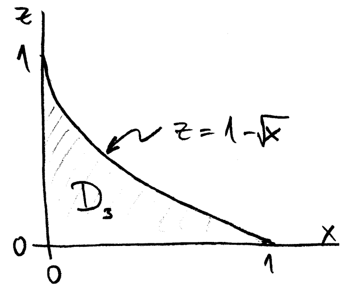

Figure 4: Region of integration in the (x,z)-plane.

$u_1(x,z) = \sqrt{x}$ and $u_2(x,z) = 1-z$ (the latter comes from the equation of the plane $z = 1-y$ from Figure 1 if we solve it with respect to y). If we want to write the integral over D_3 in the desired order of integration, we have to consider the 2-dimensional region D_3 as a region of type I, i.e., as

$$D_3 = \{(x,z) \in \mathbb{R}^2 | x \in [0,1],\ z \in [0, 1-\sqrt{x}]\}\ .$$

The equation of the line $z = 1-\sqrt{x}$ in Figure 4 is obtained by excluding y from the equations of the surfaces $y = \sqrt{x}$ and $z = 1-y$ which are boundaries of the domain E (see Figure 1). Taking all this into account, we can write

$$I_2 = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\ .$$

Method B (alternative): To do this, we can use Figure 3 again. The x-integration will go again from 0 to 1, and for a fixed value of $x \in [0,1]$, say x^*, we will have the 2-dimensional picture in Figure 3. For this fixed value of x, the variable z may vary between 0 and $(1-\sqrt{x})$ (I am omitting the star on x), so the integral becomes

$$I_2 = \int_0^1 \int_0^{1-\sqrt{x}} \int f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\ .$$

To determine the limits of the integration over y, in Figure 5 I redrew the projection onto the (y,z)-plane from Figure 3(b), but now I interpret it differently. Namely, for a given

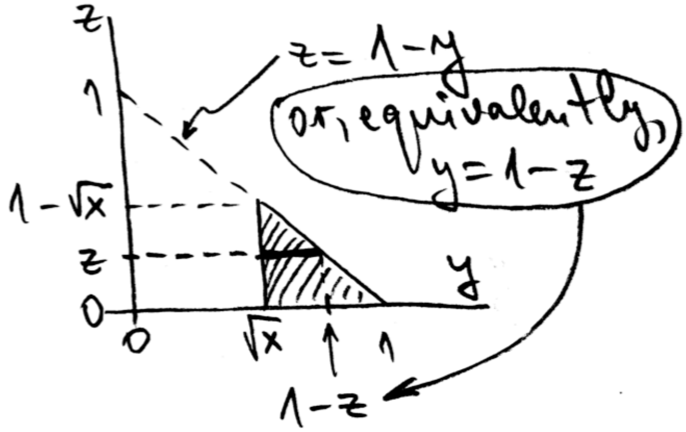

Figure 5: Projection of the plane $\{x = x^*\}$ onto the (y, z)-plane.

$z \in [0, 1 - \sqrt{x^*}]$, the allowed range for y is $[\sqrt{x}, 1 - z]$, so the integral becomes

$$I_2 = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}x \ .$$

3) Writing the integral in the form $I_3 = \int\int\int f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$**.**

Method A (from the book): We have to consider the 3-dimensional region of integration E as a region of type II, so we write the integral as

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \iint_{D_2} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, \mathrm{d}x \right] \mathrm{d}A \ ,$$

where the region of integration D_2 in the (y, z)-plane is shown in Figure 6. In this case

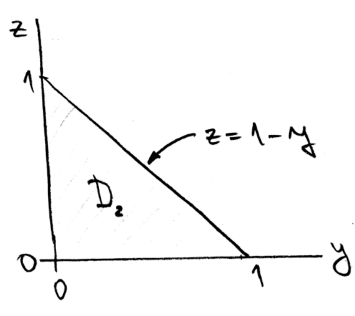

Figure 6: Region of integration in the (y, z)-plane.

$u_1(y, z) = 0$ and $u_2(y, z) = y^2$ (the latter comes from the equation of the surface $y = \sqrt{x}$

from Figure 1 if we solve it with respect to x). If we want to write the integral over D_2 in the desired order of integration, we have to consider the 2-dimensional region D_2 as a region of type II, i.e., as

$$D_2 = \{(y,z) \in \mathbb{R}^2 | z \in [0,1],\ y \in [0, 1-z]\} .$$

Taking all this into account, we can write

$$I_3 = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z .$$

Method B (alternative): The "outermost" variable z takes values from 0 to 1, so

$$I_3 = \int_0^1 \int \int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z .$$

For a fixed value of $z \in [0,1]$, the 3-dimensional and 2-dimensional pictures are shown in Figure 7 (a) and (b), resp.

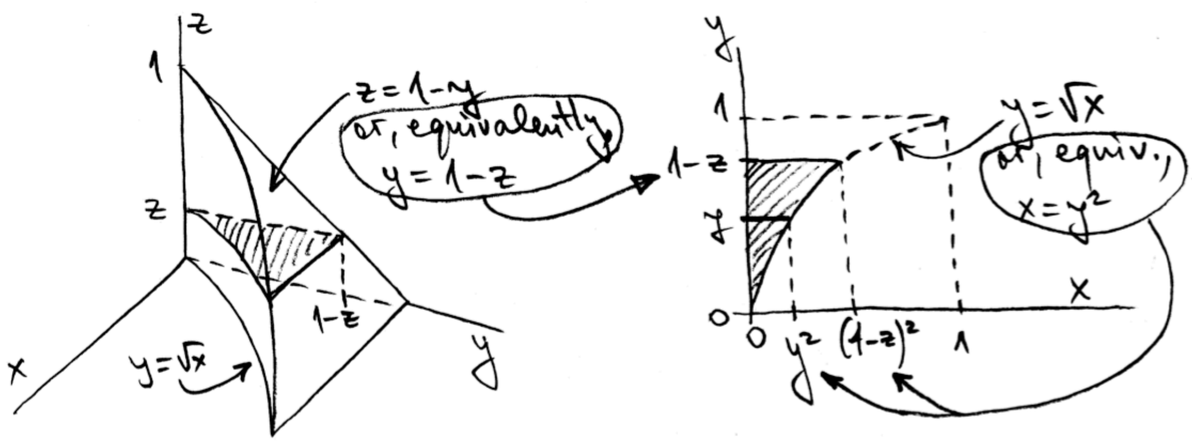

Figure 7: (a) Cross-section with the plane with fixed z; (b) projection onto the (x,y)-plane.

From Figure 7(b) we see that, for a fixed z, the value of y varies from 0 to $(1-z)$, and for fixed z and y, the value of x is allowed to vary between 0 and y^2, so

$$I_3 = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z .$$

4) Writing the integral in the form $I_4 = \int \int \int f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$.

Method A (from the book): If we consider the 3-dimensional region of integration E as a region of type III, we can write the integral as

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_{D_3} \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,\mathrm{d}y \right] \mathrm{d}A ,$$

where the region of integration D_3 in the (x, z)-plane is shown in Figure 4. In this case, as before, $u_1(x, z) = \sqrt{x}$ and $u_2(x, z) = 1 - z$. If we want to write the integral over D_3 in the desired order of integration, we have to consider the 2-dimensional region D_3 as a region of type II, i.e., as

$$D_3 = \{(x, z) \in \mathbb{R}^2 | z \in [0, 1],\ x \in [0, (1 - z)^2]\} \ .$$

Taking all this into account, we can write

$$I_4 = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z \ .$$

Method B (alternative): Using Figure 7 again, we can write

$$I_4 = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z \ .$$

As an exercise, redraw Figure 7(b), but now modify it to explain the limits of integration in I_4.

5) Writing the integral in the form $I_5 = \int \int \int f(x, y, z) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y$.

Method A (from the book): We have to consider the 3-dimensional region of integration E as a region of type II, so we write the integral as

$$\iiint_E f(x, y, z) \, \mathrm{d}V = \iint_{D_2} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) \, \mathrm{d}x \right] \mathrm{d}A \ ,$$

where the region of integration D_2 in the (y, z)-plane is shown in Figure 6. As before, $u_1(y, z) = 0$ and $u_2(y, z) = y^2$. If we want to write the integral over D_2 in the desired order of integration, we have to consider the 2-dimensional region D_2 as a region of type I, i.e., as

$$D_2 = \{(y, z) \in \mathbb{R}^2 | y \in [0, 1],\ z \in [0, 1 - y]\} \ .$$

Taking all this into account, we can write

$$I_5 = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y \ .$$

Method B (alternative): In Figure 8 (a) and (b), we show a cross-section with a plane with a fixed value of y in 3 dimensions and its projection onto the (x, z)-plane. The dashed line in Figure 8(b) is the projection of the line in $\mathbb{R}^3$ that connects the points $(0, 0, 1)$ and $(1, 1, 0)$ (see Figure 8(a)). The equation of this dashed line is $z = 1 - \sqrt{x}$, which comes from the equations of the surfaces $z = 1 - y$ and $y = \sqrt{x}$ bounding the region of integration (see Figure 1) by excluding the variable y.

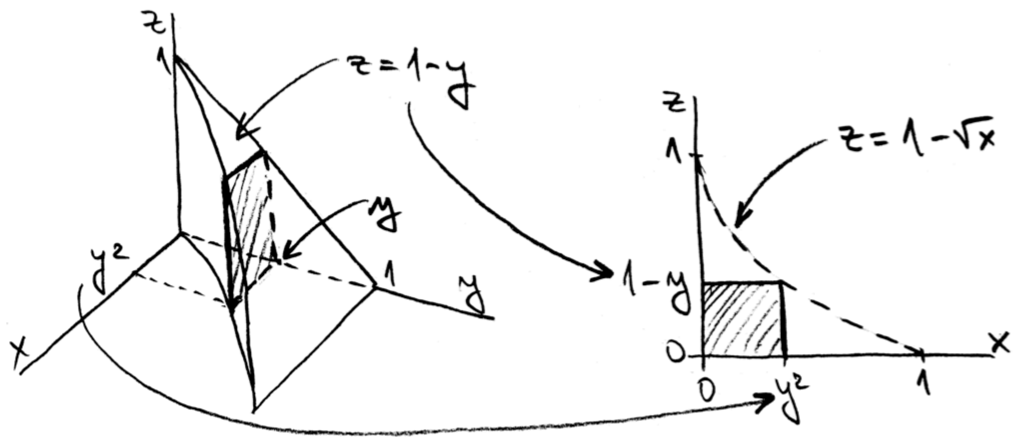

Figure 8: (a) Cross-section with the plane with fixed y; (b) projection onto the (x, z)-plane.

Looking at Figure 8(b), it is easy to write

$$I_5 = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\, \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y \ .$$

6) Writing the integral in the form $I_6 = \int \int \int f(x, y, z)\, \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y$**.**

Method A (from the book): We give only a brief sketch. Considering the 3-dimensional region of integration E as a region of type I, we write

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \iint_{D_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, \mathrm{d}z \right] \mathrm{d}A \ ,$$

where D_1 is shown in Figure 2. As before, $u_1(x, y) = 0$ and $u_2(x, y) = 1 - y$.
The double integral over D_1 should be considered as type II, i.e., as

$$D_1 = \{(x, y) \in \mathbb{R}^2 | y \in [0, 1],\ x \in [0, y^2]\} \ ,$$

which implies that

$$I_6 = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z)\, \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y \ .$$

Method B (alternative): Compared with the integral in part **5)**, we see that we only have to switch the order of the integration in the double integral over the variables x and z – see Figure 8(b). For a given value of the "outermost" variable y (between 0 and 1), the variable z takes values between 0 and $(1 - y)$; for fixed values of y and z, the variable x takes values between 0 and y^2, so that

$$I_6 = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z)\, \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y \ .$$