

MATH 2934 – Solution of Problem 15.7/7

$$\begin{aligned}
 \int_0^{\pi/2} \left[\int_0^y \left(\int_0^x \cos(x+y+z) \, dz \right) dx \right] dy &= \int_0^{\pi/2} \left[\int_0^y \left(\int_{\xi=x+y}^{2x+y} \cos \xi \, d\xi \right) dx \right] dy \\
 &\quad [\text{change of var from } z \text{ to } \xi := x + y + z, \, d\xi = dz] \\
 &= \int_0^{\pi/2} \left[\int_0^y \left(\sin(2x+y) - \sin(x+y) \right) dx \right] dy \\
 &= \int_0^{\pi/2} \left[\int_0^y \sin(2x+y) \, dx - \int_0^y \sin(x+y) \, dx \right] dy \\
 &= \int_0^{\pi/2} \left[\frac{1}{2} \int_y^{3y} \sin \eta \, d\eta - \int_y^{2y} \sin \zeta \, d\zeta \right] dy \\
 &\quad [\text{changes of var's from } x \text{ to } \eta := 2x + y, \, d\eta = 2 \, dx, \\
 &\quad \text{and from } x \text{ to } \zeta := x + y, \, d\zeta = dx] \\
 &= \int_0^{\pi/2} \left[\frac{1}{2} (\cos y - \cos 3y) - (\cos y - \cos 2y) \right] dy \\
 &= \int_0^{\pi/2} \left[-\frac{1}{2} \cos y - \frac{1}{2} \cos 3y + \cos 2y \right] dy \\
 &= -\frac{1}{2} \cdot 1 - \frac{1}{2} \left(-\frac{1}{3} \right) + 0 \\
 &= -\frac{1}{3}
 \end{aligned}$$