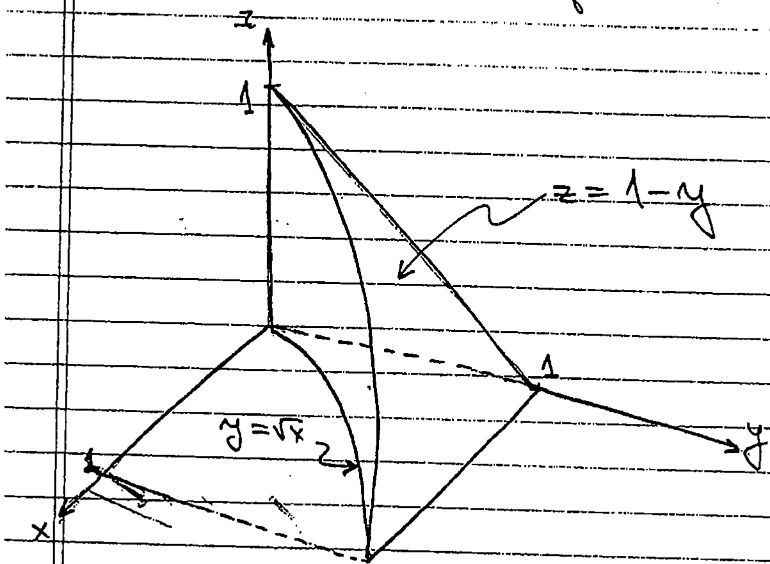


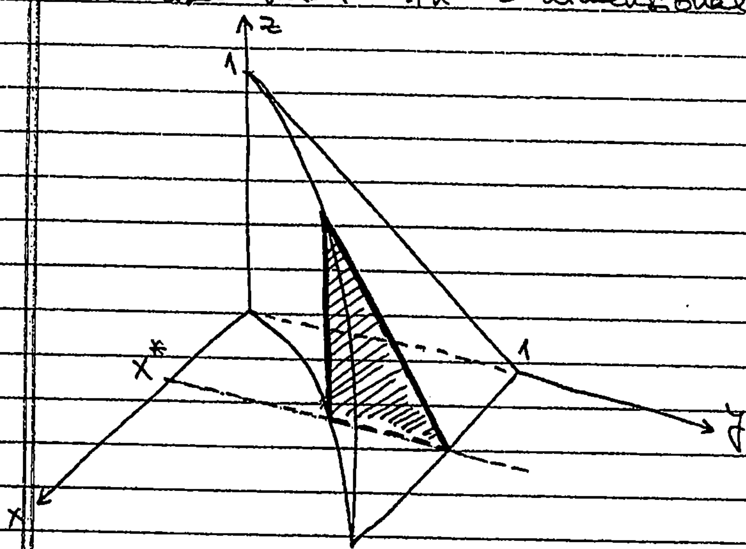
1) Writing it in the form

$$I_1 = \int\int\int f(x,y,z)\, dz\, dy\, dx = \int dx \int dy \int dz\, f(\ldots).$$

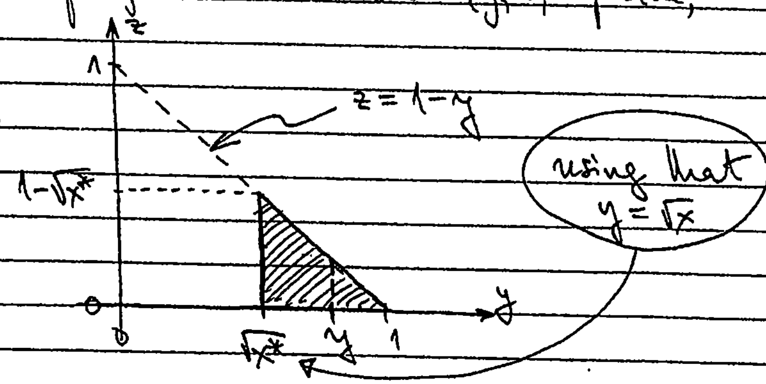
I will use the second form of writing the integral. Clearly the "outside" integration goes for x from 0 to 1, so

$$I_1 = \int_0^1 dx \int dy \int dz\, f(\ldots).$$

For a fixed value of $x \in [0,1]$, the cross-section of the body in the figure with the plane $\{x = (\text{the chosen value } x^*)\}$, looks like this: in 3-dimensional picture,



and projected onto the (y,z)-plane,



(Check: When $x^*=1$, this triangle disappears; when $x^*=0$, we get the "Big" triangle in the 2-dim picture).

So the second integration (over y) goes from $\sqrt{x^*}$ to 1:

$$I_1 = \int_0^1 dx^* \int_{\sqrt{x^*}}^1 dy \int dz \;\; f(\;)$$

Finally, for a fixed value of y, the variable z is allowed to vary between 0 and $1-y$ (see the 2-dim picture above), so

$$I_1 = \int_0^1 dx \int_{\sqrt{x}}^1 dy \int_0^{1-y} dz\, f(x,y,z).$$

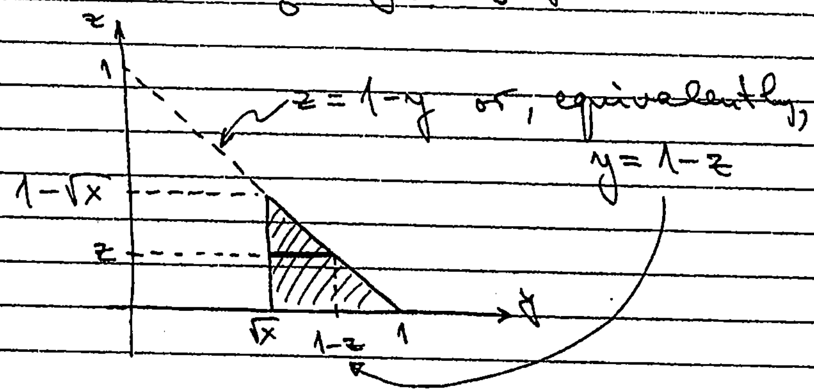
Here I omitted the star on the x, which was used only temporarily to make things clearer; from now on I will not use such temporary notations.

2) It is easy to use the above pictures in order to write the integral in the form

$$I_2 = \int dx \int dz \int dy \; f.$$

The integration over x will go again from 0 to 1 and for a fixed value of x in $[0,1]$, we will have the 2-dim picture above. For this fixed value of x, the variable z may vary between 0 and $1-\sqrt{x}$ (I am omitting the star), so the integral becomes

$$I_2 = \int_0^1 dx \int_0^{1-\sqrt{x}} dz \int dy\, f(\,).$$



For a given $z \in [0, 1-\sqrt{x}]$, the allowed range for y is $[\sqrt{x}, 1-z]$, so the integral becomes

$$I_2 = \int_0^1 dx \int_0^{1-\sqrt{x}} dz \int_{\sqrt{x}}^{1-z} dy\, f(x,y,z).$$

3) Now I will take z to be the "outside" variable, and will write the integral in the form

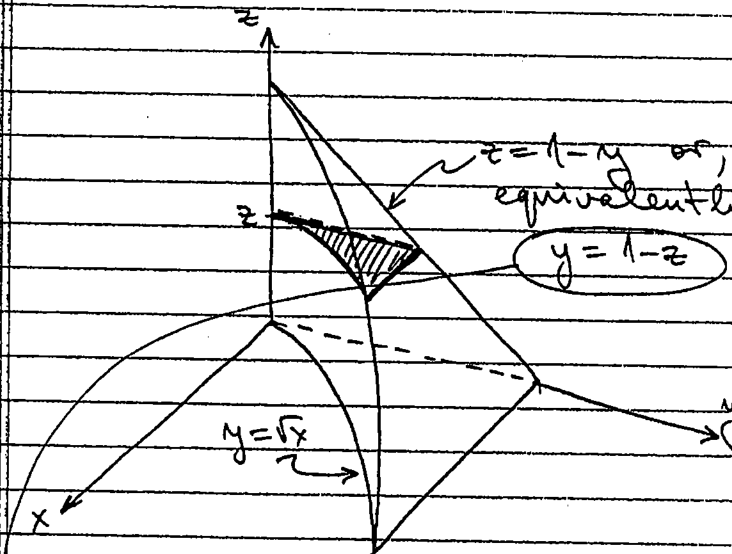
$$I_3 = \int dz \int dy \int dx\, f(\ldots).$$

The variable z takes all values between 0 and 1, so

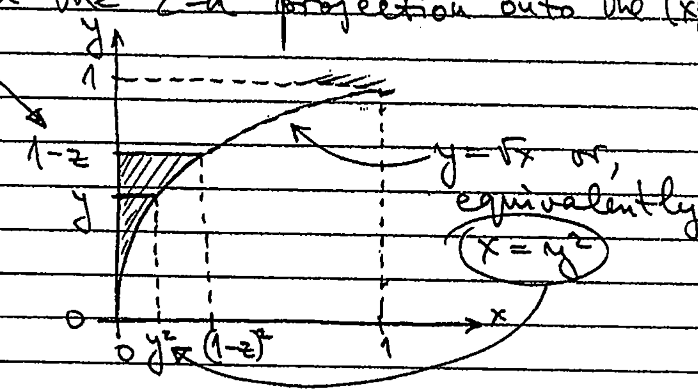
$$I_3 = \int_0^1 dz \int dy \int dx\, f.$$

For a fixed value of $z \in [0,1]$, we have

the following 3-d picture:



and the 2-d projection onto the (xy) plane:



For a fixed $z \in [0,1]$, y varies from 0 to $1-z$, and for fixed z and y x varies from 0 to y^2, so we get

$$I_3 = \int_0^1 dz \int_0^{1-z} dy \int_0^{y^2} dx \; f(\dots).$$

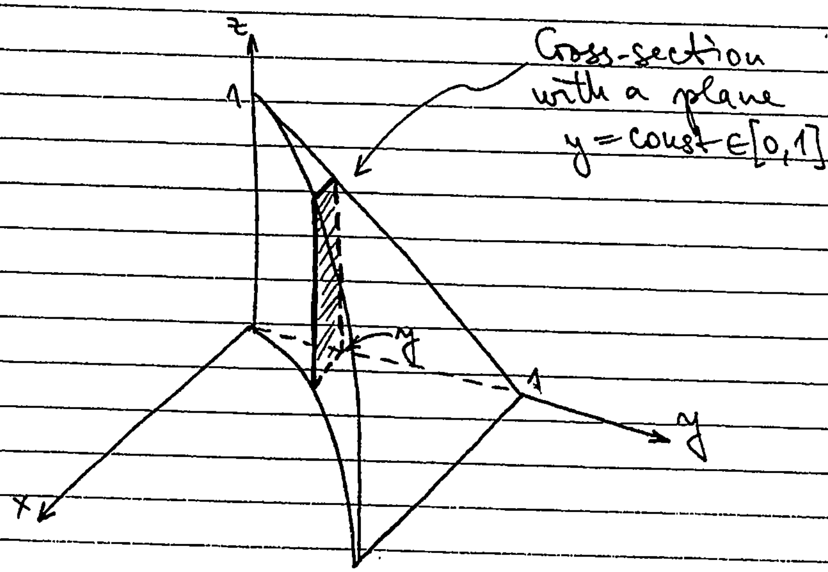
4) Using the same pictures as in part 3), one can write

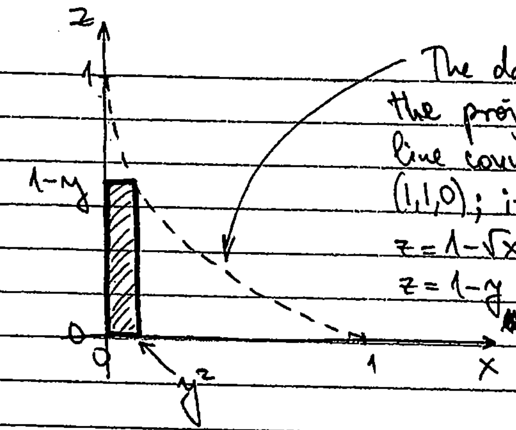
$$I_4 = \int_0^1 dz \int_0^{(1-z)^2} dx \int_{\sqrt{x}}^{1-z} dy \; f(\dots).$$

Please draw a 2-d picture that clarifies the integrations over x and y as an exercise.

5) Now we want to write

$$I_5 = \int dy \int dz \int dx \; f(\dots)$$





The dashed line here is the projection of the 3-d line connecting $(0,0,1)$ and $(1,1,0)$; its equation is $z = 1-\sqrt{x}$, which comes from $z = 1-y$ and $y = \sqrt{x}$.

It is easy to write

$$I_5 = \int_0^1 dy \int_0^{1-y} dz \int_0^{y^2} dx \; f(\dots).$$

6) Finally we reverse the order of the internal integration in I_5, which is easy because the projection of the cross-section in part 5) onto the (x,z)-plane is a rectangle — what does this fact imply abouts the limits of integration over x and z at fixed y? The result is

$$I_6 = \int_0^1 dy \int_0^{y^2} dx \int_0^{1-y} dz \; f(\dots).$$