PRINT NAME: SOLUTIONS

Honors Calculus II [2423-001] Midterm II

Q1]...[20 points] Evaluate the following expressions and limits. Since you don't have a calculator, square roots, fractions etc. are allowed in your answers.

- $\log_2(\frac{1}{8})$

  Answer: Since $1/8 = 8^{-1} = (2^3)^{-1} = 2^{-3}$ we have that $\log_2(\frac{1}{8}) = -3$.

- $\cos(\sin^{-1}(-0.2))$

  Answer:
$$\cos(\sin^{-1}(-0.2)) = \sqrt{1-(-0.2)^2} = \sqrt{1-0.04} = \sqrt{0.96}$$

- $\tan(\cos^{-1}(0.3))$

  Answer:
$$\tan(\cos^{-1}(0.3)) = \frac{\sin(\cos^{-1}(0.3))}{\cos(\cos^{-1}(0.3))} = \frac{\sqrt{1-(0.3)^2}}{0.3} = \frac{10\sqrt{0.91}}{3} = \frac{\sqrt{91}}{3}$$

- $\lim_{x\to\infty} \frac{x^{1,000,000}}{(1.0000001)^x}$

  Answer: 0 since exponentials in x (with base greater than 1) always grow faster than any power of x (seen in class).

- $\lim_{x\to\infty} \frac{\ln(x)}{x^{0.1}}$

  Answer: 0 since $\ln x$ always grows more slowly than any positive power of x (again, seen in class).

Q2]...[20 points] Compute the following derivative.

$$\frac{d}{dx}\,(\sin x)^{x^2}$$

Use logarithmic differentiation.

$$\ln y = x^2 \ln(\sin x)$$

and so taking derivatives w.r.t x of both sides gives

$$\frac{1}{y} y' = 2x \ln(\sin x) + x^2 \frac{1}{\sin x} \cos x$$

Thus

$$y' = (\sin x)^{x^2} \left[2x \ln(\sin x) + x^2 \cot x\right]$$

Compute the following limit.

$$\lim_{x \to 0} x^{\sqrt{x}}$$

Take logs to begin with.

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \sqrt{x} \ln(x) = \lim_{x \to 0} \frac{\ln x}{x^{-1/2}}$$

By the ∞/∞ form of l'Hôpital's rule we get

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{1/x}{-1/2x\sqrt{x}} = \lim_{x \to 0} -2\sqrt{x} = 0$$

Thus

$$\lim_{x \to 0} y = e^{\lim_{x \to 0} \ln y} = e^0 = 1$$

Q3]...[20 points] Find the inverse of the following function (show all your work).

$$y = \tanh x$$

Hint: I'll tell you that $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Step 1. Interchange x and y.

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Step 2. Solve for new y in terms of new x.

$$xe^y + xe^{-y} = e^y - e^{-y}$$

Multiply across by e^y to get

$$xe^{2y} + x = e^{2y} - 1$$

or

$$1 + x = e^{2y}(1 - x)$$

Thus we get

$$e^{2y} = \frac{(1+x)}{(1-x)}$$

and so we take logs and divide by 2 to get

$$y = \frac{1}{2}\ln\left[\frac{(1+x)}{(1-x)}\right]$$

Suppose that a function f has an inverse g, and that $f(2) = 3$, $f'(2) = 4$ and $f''(2) = 7$. Find $g'(3)$.

We know that

$$g'(x) = \frac{1}{f'(g(x))}$$

which, in our case, gives

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(2)} = \frac{1}{4}$$

Find $g''(3)$.

We recall from class (or derive again) that

$$g''(x) = \frac{d}{dx}g'(x) = \frac{d}{dx}\frac{1}{f'(g(x))} = \frac{-1}{(f'(g(x)))^2}f''(g(x))g'(x) = \frac{-f''(g(x))}{(f'(g(x)))^3}$$

In our case we get,

$$g''(3) = \frac{-f''(2)}{(f'(2))^3} = \frac{-7}{64}$$

Q4]...[20 points] Let

$$h(x) = \frac{\ln x}{x} \qquad \text{defined for } x > 0$$

- Show that $h'(x) > 0$ for $0 < x < e$.

  We have

  $$h'(x) = \frac{x\frac{1}{x} - 1\ln(x)}{x^2} = \frac{1 - \ln x}{x^2}$$

  Thus

  $$h'(x) > 0$$

  if and only if

  $$1 - \ln x > 0$$

  and this is true if and only if

  $$0 < x < e$$

- Show that $h'(x) < 0$ for $x > e$.

  Again,

  $$h'(x) < 0$$

  if and only if

  $$1 - \ln x < 0$$

  and this is true if and only if

  $$x > e$$

- Is the point $(e, h(e))$ a local max or a local min for h?

  It is a local max.

- Which is larger; e^π or π^e?
  Hint: this is related to the other parts of the question!

  By the previous part we know that

  $$\frac{\ln e}{e} > \frac{\ln \pi}{\pi}$$

  This gives (on multiplying across by the positive number $e\pi$)

  $$\pi \ln e > e \ln \pi$$

  or

  $$\ln(e^\pi) > \ln(\pi^e)$$

  Taking exponentials of both sides (and remembering that e^x is an increasing function) gives

  $$e^\pi > \pi^e$$

Q5]...[20 points] Evaluate the following integrals.

$$\int \frac{dx}{2x^2+3}$$

This gave people more trouble than I expected! You notice that there is no xdx on the numerator, so we can't expect a substitution of the form $u = x^2$ or $u = 2x^2 + 3$ to work. Also, you see (after a few attempts) that integration by parts is leading nowhere. So it must be something we already know: let's see.....what derivative involves reciprocals, x^2 and $+$-signs .....yes! the derivative of $\tan^{-1}(x)$.

In fact we remember from class (with the memory jog on the sheet at the end of the exam) that

$$\frac{d}{dx}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{x^2+a^2}$$

Thus, our integral is

$$\frac{1}{2}\int \frac{dx}{x^2+(\sqrt{3/2})^2} = \frac{1}{2}\frac{\sqrt{2}}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{3}}\right) + C$$

$$\int e^{2x}\sin(3x)\,dx$$

Use integration by parts (with $u = \sin(3x)$ and $dv = e^{2x}dx$) to get

$$\int e^{2x}\sin(3x)\,dx \;=\; \sin(3x)\frac{e^{2x}}{2} - \frac{1}{2}\int e^{2x}3\cos(3x)\,dx \;=\; \frac{\sin(3x)e^{2x}}{2} - \frac{3}{2}\int e^{2x}\cos(3x)\,dx$$

This last integral is similar to the original, so try integration by parts on this last integral (with $u = \cos(3x)$ and $dv = e^{2x}dx$) to get

$$\int e^{2x}\sin(3x)\,dx \;=\; \frac{\sin(3x)e^{2x}}{2} - \frac{3}{2}\left[\frac{\cos(3x)e^{2x}}{2} - \frac{-3}{2}\int \sin(3x)e^{2x}\,dx\right]$$

Simplifying, gives

$$\int e^{2x}\sin(3x)\,dx \;=\; \frac{\sin(3x)e^{2x}}{2} - \frac{3\cos(3x)e^{2x}}{4} - \frac{9}{4}\int e^{2x}\sin(3x)\,dx$$

Now, the last integral is identical to the first! Taking it to the left side, gives

$$\frac{13}{4}\int e^{2x}\sin(3x)\,dx \;=\; \frac{\sin(3x)e^{2x}}{2} - \frac{3\cos(3x)e^{2x}}{4}$$

and so

$$\int e^{2x}\sin(3x)\,dx \;=\; \frac{4}{13}\left[\frac{\sin(3x)e^{2x}}{2} - \frac{3\cos(3x)e^{2x}}{4}\right] + C$$

Remark: You could have done the integration by parts with $u = e^{2x}$ and $dv = \sin(3x)dx$ etc.) You should get the same answer.

Bonus Questions Compute the limit

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$$

Take logs first.

$$\lim_{x\to 0}\ln y \;=\; \lim_{x\to 0}\left[\frac{\ln\left(\frac{\sin x}{x}\right)}{x^2}\right]$$

which is equal (by l'Hôpital's rule in the 0/0 case) to

$$\lim_{x\to 0}\left[\frac{\frac{x}{\sin x}\,\frac{x\cos x-1\sin x}{x^2}}{2x}\right] \;=\; \lim_{x\to 0}\frac{x\cos x-\sin x}{2x^2\sin x}$$

Again, by l'Hôpital's rule (0/0-case) this is in turn equal to

$$\lim_{x\to 0}\frac{\cos x - x\sin x-\cos x}{4x\sin x+2x^2\cos x} \;=\; \lim_{x\to 0}\frac{-\sin x}{4\sin x+2x\cos x}$$

Again(!), by l'Hôpital's rule (0/0-case) this is equal to

$$\lim_{x\to 0}\frac{-\cos x}{4\cos x+2\cos x-2x\sin x} \;=\; \frac{-1}{6}$$

Finally,

$$\lim_{x\to 0} y = e^{-1/6}\,.$$

If $f''(x)$ is continuous, show that

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} \;=\; f''(x)$$

Use l'Hôpital's rule (in the 0/0-case) to say that the LHS is equal to

$$\lim_{h\to 0}\frac{f'(x+h)-0+f'(x-h)(-1)}{2h} \;=\; \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

Again, l'Hôpital's rule (0/0-case) tells us that this limit is the same as

$$\lim_{h\to 0}\frac{f''(x+h)-f''(x-h)(-1)}{2} \;=\; \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}$$

Now, continuity of f'' tells us that as $h\to 0$ then $f''(x\pm h)\to f''(x)$, and so this last limit is just

$$\frac{f''(x)+f''(x)}{2} \;=\; f''(x)$$

And it's done!

Handy formulae

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$
- $\cosh(x) = \frac{e^x + e^{-x}}{2}$
- $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- $\frac{d\sin^{-1}(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d\cos^{-1}(x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2}$
- $\frac{d\sinh^{-1}(x)}{dx} = \frac{1}{\sqrt{1+x^2}}$
- $\frac{d\cosh^{-1}(x)}{dx} = \frac{1}{\sqrt{x^2-1}}$
- $\frac{d\tanh^{-1}(x)}{dx} = \frac{1}{1-x^2}$