PRINT NAME: <u>SOLUTIONS</u>

Honors Calculus II [2423-001] Midterm I

For full credit, please show all your work.

Q1]...[12 points] Evaluate the following limit of Riemann sums by first converting to a definite integral, and then evaluating the integral.

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{n}\sqrt{1+\frac{i}{n}}\right)$$

Let subdivision points be $x_i = 1+i/n$. As $n\to\infty$ these points range between a lower limit of $1+0=1$ and an upper limit of $1+n/n=2$. The widths are all equal to $1/n$. Thus the limit of Riemann sums becomes the following definite integral.

$$\int_1^2 \sqrt{x}\,dx \;=\; \frac{2}{3}x^{3/2}\Big|_1^2 \;=\; \frac{2}{3}\left(2\sqrt{2}-1\right)$$

Q2]...[11 points] Compute the following derivative.

$$\frac{d}{dx}\left(\int_x^{x^3}\cos(t^2)\,dt\right)$$

By the fundamental theorem and the chain rule we know that

$$\frac{d}{dx}\left(\int_{l(x)}^{k(x)} f(t)\,dt\right) \;=\; k'(x)f(k(x)) - l'(x)f(l(x))\,.$$

In our case we have $f(t)=\cos(t^2)$, $k(x)=x^3$ and $l(x)=x$, and so we get

$$\frac{d}{dx}\left(\int_x^{x^3}\cos(t^2)\,dt\right) \;=\; 3x^2\cos(x^6)-\cos(x^2)\,.$$

Q3]...[33 points] By interpreting things geometrically (that is *areas, odd-even functions,* etc...), find the values of the following integrals. You should **not** do anti-differentiation and evaluation!

$$\int_1^3 |x-2|\,dx$$

The graph of $y=|x-2|$ is the line $y=x-2$ when $x\geq 2$ and is the line $y=2-x$ when $x\leq 2$. The region between this graph and the x-axis between the vertical lines $x=1$ and $x=3$ consists of two half squares (of edge length 1). Thus the value of our integral is the positive area under this graph which is $1/2+1/2=1$.

$$\int_0^{\sqrt{5}} \sqrt{5-x^2}\, dx$$

The graph of $y = \sqrt{5-x^2}$ is the upper half of the circle $(x^2+y^2 = 5 = (\sqrt{5})^2)$ of radius $\sqrt{5}$ centered on the origin. The integral represents the area of the quarter of this circle which resides in the first quadrant, and so has value equal to $\pi(\sqrt{5})^2/4 = 5\pi/4$.

$$\int_{-\pi}^{\pi} (2+\sin(x^3))\, dx$$

This integral splits as a sum of two integrals, with integrands 2 and $\sin(x^3)$ respectively. The first integral represents the area of a rectangle with base from $-\pi$ to π (and so of length 2π) and height 2, and so is $2(2\pi) = 4\pi$. The second integral has a value of 0 since the function $\sin(x^3)$ is an odd function $\sin((-x)^3) = -\sin(x^3)$ and the region is symmetric about the origin. Thus the value of the original integral is $4\pi + 0 = 4\pi$.

Q4]...[33 points] This question asks you to *write down* definite integrals corresponding to volumes of revolution. *You do* **not** *have to evaluate the integrals.* You should draw a picture in each case.

- The volume obtained by revolving the region between the graphs of $y = x^2$ and $y = \sqrt{x}$ about the x-axis. Use the **cylindrical shell** method.

  Since we are to use the cylindrical shell method, and we are rotating about the x-axis, then we should use horizontal strips. These will have thickness dy and will begin on the graph $y = \sqrt{x}$ (that is $x = y^2$) and will end on the graph $y = x^2$ (that is $x = \sqrt{y}$). Thus, we get

  $$dV \;=\; 2\pi(radius)(length)(thickness) \;=\; 2\pi(y)(\sqrt{y}-y^2)(dy)\,.$$

  Finally, noting that the horizontal strips are parameterized by y starting at 0 and ending at 1, we get

  $$V \;=\; \int_0^1 2\pi y(\sqrt{y}-y^2)\, dy\,.$$

- The volume obtained by revolving the region between the graphs of $y = x^2$ and $y = \sqrt{x}$ about the x-axis. Use the **washer** method.

  Since we have to use the washer method, and since we are rotating about the x-axis, then we should use vertical strips. These will have thickness dx and will determine an inner radius of x^2 and an outer radius of $\sqrt{x}$. Thus

  $$dV \;=\; \pi[(rad_o)^2 - (rad_i)^2](thickness) \;=\; \pi[x - x^4]dx$$

  and so we get

  $$V \;=\; \int_0^1 \pi(x - x^4)\, dx\,.$$

- The volume obtained by revolving the region between the graphs of $y = x^3$, $x = 0$ and $y = 1$ about the y-axis. Use the **disk** method.

Since we are rotating about the y-axis, and since we are using the disk method, then we should use horizontal strips. They will have thickness dy and length equal to $x = y^{1/3}$. Thus

$$dV \;=\; \pi(rad)^2(thickness) \;=\; \pi(y^{1/3})^2(dy)$$

and so

$$V \;=\; \int_0^1 \pi y^{2/3}\, dy\,.$$

Q5]...[11 points] Use the **method of substitution** to evaluate the following definite integral. Show clearly what substitution you are making, and show all the details of your work.

$$\int_0^a x\sqrt{a^2 - x^2}\, dx$$

Let $u = a^2 - x^2$. Note that when $x = 0$ then $u = a^2 - 0^2 = a^2$. Also, when $x = a$ then $u = a^2 - a^2 = 0$. Furthermore,

$$du \;=\; -2xdx \qquad \text{or} \qquad xdx \;=\; -du/2$$

and the original integral becomes

$$-\frac{1}{2}\int_{a^2}^0 \sqrt{u}\, du \;=\; \frac{1}{2}\int_0^{a^2} \sqrt{u}\, du \;=\; \frac{1}{2}\frac{2}{3}u^{3/2}\Big|_0^{a^2} \;=\; \frac{a^3}{3}\,.$$

Bonus Question Use the substitution $u = \pi - x$ to show that the following definite integrals are equal for any continuous function f.

$$\int_0^\pi x f(\sin x)\, dx \;=\; \frac{\pi}{2}\int_0^\pi f(\sin x)\, dx$$

As was suggested, we let $u = \pi - x$. Thus $du = -dx$, when $x = 0$ then $u = \pi - 0 = \pi$, and when $x = \pi$ then $u = \pi - \pi = 0$. Thus the first integral (on the LHS) becomes

$$LHS \;=\; -\int_\pi^0 (\pi - u) f(\sin(\pi - u))\, du$$

which looks like a mess!

However, we know (should know!) from trigonometry that

$$\sin(\pi - u) \;=\; -\sin(-u) \;=\; \sin(u)\,.$$

The first equality comes from the fact that when we add $\pm\pi$ to the input of a sine (or cosine) function, then we change the sign of the output, while the second equality is just the fact that sine is an odd function. This is the first nontrivial part of this answer. Thus we can simplify the integrand a bit to get

$$LHS \;=\; \int_0^\pi (\pi - u) f(\sin(u))\, du$$

which looks a lot better.

In fact, we can expand this out a bit to get

$$LHS \;=\; \int_0^\pi \pi f(\sin(u))\,du \;-\; \int_0^\pi u f(\sin(u))\,du$$

and we notice something very, very nice indeed (in fact, the whole point of this exercise!). Namely, the second term above is (by replacing the dummy variable of integration u by another one x say) identical to the original LHS. This is the second nontrivial part of the question. Thus we have

$$LHS \;=\; \pi \int_0^\pi f(\sin(u))\,du \;-\; LHS$$

or

$$2(LHS) \;=\; \pi \int_0^\pi f(\sin(u))\,du$$

Finally, we get

$$LHS \;=\; \frac{\pi}{2} \int_0^\pi f(\sin x)\,dx \;=\; RHS$$

which is what we needed to show (again, we replaced dummy variables of integration; x for u).