MATH 2513-001 Discrete Mathematics Fall 2005

Cardinality of Infinite Sets

1. **Definition.** Two sets A and B have the *same cardinality*, denoted by $|A| = |B|$, if there exists a bijection $f : A \to B$. This is the *one cow, one sheep* definition of cardinality.

   The cardinality of a set A is *less than or equal to* the cardinality of a set B, denoted by $|A| \leq |B|$, if there exists an injection $f : A \to B$.

2. **Definition.** A set is said to be *countably infinite* if it has the same cardinality as the positive integers $\mathbb{Z}^+$.

   A set is said to be *countable* if it is finite, or countably infinite.

3. **Examples.** The following sets are all countably infinite; the set of even integers, the set of odd integers, the set of all integers, $\mathbb{Z}^+ \times \mathbb{Z}^+$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, an n-fold cartesian product $\mathbb{Z} \times \cdots \times \mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{Q}$.

4. **Examples.** Show that if A and B are disjoint, countably infinite sets, then $A \cup B$ is also countably infinite. Show that the same result holds if A and B are not necessarily disjoint.

   Show that if $A_1, \ldots, A_n$ are n mutually disjoint, countably infinite sets, then $A_1 \cup \cdots \cup A_n$ is also countably infinite. Show that the same result holds without the disjointness assumption.

   Show that if $\{A_n\}_{n \in \mathbb{Z}^+}$ is a countable collection of countable sets, then the union

   $$\bigcup_{n=1}^{\infty} A_n$$

   is countable.

5. **Theorem.** If $|A| \leq |B|$, and B is countable, then A is also countable.

6. **Examples.** The following sets of real numbers all have the same cardinality; $\mathbb{R}$, $(0, \infty)$, $(-\pi/2, \pi/2)$, $(0, \pi)$, $(0, 1)$, (a, b) for any $a < b$, $[a, b)$, $(a, b]$, $[a, b]$.

7. **Theorem (baby Cantor).** The set $(0, 1)$ is not countable. In fact, there is no surjective map $\mathbb{Z}^+ \to (0, 1)$.

8. **Definition.** An infinite set which is not countable is said to be *uncountably infinite*. The set $\mathbb{R}$ is uncountably infinite.

9. **Examples.** Show that the set of all irrational numbers is uncountably infinite. [Hint: points 3, 4, 6 and 7 all combine to give an argument by contradiction]

10. **Definiton.** A real number r is called *algebraic* if it is the solution of a polynomial with integer coefficients.

11. **Example.** Show that $\sqrt{2}$ is an algebraic number. Show that every rational number is algebraic. Show that the set of all algebraic numbers is countably infinite. Therefore, its compliment (the set of *transcendental numbers*) is uncountably infinite.

12. **Theorem (grown up Cantor).** For any set A we have $|A| \leq |\mathcal{P}(A)|$ but $|A| \neq |\mathcal{P}(A)|$.

13. **Corollary.** There is an infinite hierarchy of cardinalities of infinite sets.

    $$|\mathbb{Z}^+| \ < \ |\mathcal{P}(\mathbb{Z}^+)| \ < \ |\mathcal{P}(\mathcal{P}(\mathbb{Z}^+))| \ < \ \cdots$$

14. **Theorem (Schröder-Bernstein).** Let A and B be sets. If

$$|A| \leq |B| \qquad \text{and} \qquad |B| \leq |A|$$

then $|A| = |B|$.

15. **Example.** Trace through the proof of Schröder-Bernstein above in the case that $A = B = \mathbb{Z}^+$ and the two injective maps are $n \mapsto 2n$ and $n \mapsto 3n$.

16. **Theorem.** There is an injective map $\mathcal{P}(\mathbb{Z}^+) \to (0, 1)$.

    Theorem. There is an injective map $(0, 1) \to \mathcal{P}(\mathbb{Z}^+)$.

17. **Corollary.** $\mathcal{P}(\mathbb{Z}^+)$ has the same cardinality as $\mathbb{R}$.

    This cardinality is called *the power of the continuum*. It is an axiom of the foundations of mathematics (which is known to be independent of the remaining axioms), that there are no sets with cardinality strictly between $\mathbb{Z}^+$ and $\mathbb{R}$. This axiom is known as *the continuum hypothesis*.

18. **Exercise.** Show that $\mathbb{R}^2$ and $\mathbb{R}$ have the same cardinality. [Hint: We already know that there are many injective maps $\mathbb{R} \to \mathbb{R}^2$. If only we could find an injective map going the other direction, we'd be done by Schröder-Bernstein...]

19. **Weird scenes inside the goldmine...[The Cantor Set].** Let $A_0 = [0, 1]$, and for each $n \geq 1$ define

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

and finally define the Cantor set, C, to be the intersection

$$C = \bigcap_{n \in \mathbb{Z}^+} A_n$$

   (a) Show that C is uncountable.

   (b) Show that C has *no length!*. [Hint: C is a subset of each A_n. Show that the total length of a given A_n is $(2/3)^n$]

Remarks. The *generalized continuum hypothesis* states that there are no sets with cardinality strictly between any of the sets listed in Corollary 13 above. It is independent of the axioms of set theory.

The result of Exercise 18 is remarkable. It is hard to imagine a bijective map from $\mathbb{R}$ to $\mathbb{R}^2$, even a *surjective* map seems unreasonable. There is a beautiful map due to Peano (called the Peano curve) which is a *continuous map* with domain the interval $[0, 1]$ and range the 2-dimensional square $[0, 1]^2$. If this sounds *way out there* but still strangely compelling, you might want to consider taking a *Topology* or an *Analysis* course. In these courses, you would also learn lots more about the Cantor set.