Problems for Section 1.4.

A

In problems 1 through 6, tell whether or not the equation has a solution.

1. $3x + 5y = 50{,}001$
2. $6x + 9y = 60{,}001$
3. $21x - 14y = 10{,}000$
4. $-12x + 42y = 366$
5. $529x + 2024y = 391$
6. $851x + 1147y = 481$

Use the Euclidean algorithm method to find one solution to the equations in problems 7 through 12.

7. $7x + 20y = 3$
8. $8x + 21y = 5$
9. $66x + 51y = 300$
10. $65x + 50y = 300$
11. $200x - 300y = 400$
12. $55x + 200y = -100$

In problems 13 through 19, find all solutions with x and y positive.

13. $5x + 6y = 100$
14. $6x + 7y = 200$
15. $6x + 8y = 120$
16. $9x + 6y = 150$
17. $121x + 561y = 13{,}200$
18. $169x + 663y = 2340$
19. $621x + 1026y = 49{,}194$

B

20. If $abc \neq 0$, is it possible for $ax + by = c$ to have infinitely many solutions in positive integers?
21. For what triples a, b, c is it true that for each integer x there is an integer y such that $ax + by = c$?
22. A girl spent \$100.64 on posters. Some cost \$4.98 and some \$5.98. How many did she buy?
23. A man bought three dozen oranges and two dozen apples. His change from a \$10 bill was \$1.96. One orange costs more than 10 cents and one apple more than 15 cents. How much does an orange cost?
24. A roadside stand bought 11 large baskets of eggs from a farmer and sold 39 small baskets of eggs, which hold fewer than a dozen. There were 19 eggs left over. How many eggs does a large basket hold?
25. Farmer Jones owes Farmer Brown \$10. Neither has any cash, but Jones has 14 cows, which he values at \$185 each. He suggests paying his debt in cows, with Brown making change by giving Jones some of his pigs at \$110 each. Is this possible, and how?
26. Brown wants his \$10, but says his pigs are worth \$111 each. Is the deal still possible, and how?
27. Jones offers to trade his cows for \$184 each if Brown accepts \$100 per pig. Now what?
28. Harry bought some 39-cent pens. He paid with a \$10 bill, getting 42 cents in coins in change (plus maybe some bills). How many did he buy?
29. The same Harry as in the last problem bought 28 more expensive pens at another store, receiving 44 cents change in coins from a \$20 bill, plus perhaps some bills. How much were the pens?

1. $-3, -2, -1, 1, 2, 3$
2. $0, 3, 6, 9, 12, 15$
3. $0, 5, 10, 15, 20, 25$
4. $0, 1, 2, 3, 4, 5, 6$
5. $1, 2, 4, 8, 3, 5$
6. $3, 8, 7, 18, 23, 22$
7. $7, 14, 21, 28, 35$
8. $5, -5, 4, -4, 3, 60$

In the next eight problems, tell whether each list of values of x forms a complete solution to the given congruence.

9. $x = 4$; $2x \equiv 8 \pmod 6$
10. $x = 4, 10$; $2x \equiv 8 \pmod 6$
11. $x = 4$; $2x \equiv 8 \pmod 9$
12. $x = 4, 10$; $10x \equiv 4 \pmod{12}$
13. $x = 0$; $6x \equiv 4 \pmod{12}$
14. $x = 3, 7$; $3x \equiv 9 \pmod{12}$
15. $x = 3$; $x^2 \equiv 2 \pmod 7$
16. $x = 7$; $x^2 + x \equiv 1 \pmod 5$

In the next eight problems, tell the number of elements in a complete solution.

17. $5x \equiv 100 \pmod{55}$
18. $11x \equiv 14 \pmod{23}$
19. $91x \equiv 169 \pmod{143}$
20. $91x \equiv 169 \pmod{140}$
21. $1001x \equiv 143 \pmod{99}$
22. $48x \equiv 128 \pmod{1000}$
23. $x^2 + x + 1 \equiv 0 \pmod{14}$
24. $x^2 + x + 1 \equiv 0 \pmod{91}$

In the next 10 problems, give the least complete solution to the congruence.

25. $6x \equiv 2 \pmod 8$
26. $36x \equiv 30 \pmod{42}$
27. $25x \equiv 100 \pmod{35}$
28. $143x \equiv 169 \pmod{110}$
29. $27x \equiv -18 \pmod{15}$
30. $51x \equiv 0 \pmod{17}$
31. $3x \equiv 18 \pmod{18}$
32. $253x \equiv 341 \pmod{299}$
33. $165x \equiv 84 \pmod{221}$
34. $441x \equiv 465 \pmod{640}$

35. Find all x, $0 \le x \le 9$, such that $5x \equiv 15 \pmod{10}$.
36. Find all x, $|x| < 5$, such that $5x \equiv 20 \pmod{10}$.
37. Find all x, $100 \le x < 110$, such that $6x \equiv 2 \pmod{10}$.
38. Find all x, $100 \le x < 110$, such that $6x \equiv 9 \pmod{15}$.

B

In the next three problems, you may want to invoke the **pigeon-hole principle**: *if $n+1$ objects are put into n boxes, then some box must contain more than one object. Assume $b > 0$.*

39. Show that if S and T are two congruence classes modulo b, then either $S = T$ or else S and T have no elements in common.
40. Show that if S has b elements and no two elements of S are congruent modulo b, then S is a complete residue system modulo b.

5. $x \equiv 2 \pmod 3$
   $x \equiv 2 \pmod 5$
   $x \equiv 2 \pmod 7$
   $0 < x < 200$

6. $x \equiv 3 \pmod 4$
   $x \equiv 3 \pmod 5$
   $x \equiv 3 \pmod 7$
   $0 < x < 200$

7. $x \equiv 1 \pmod 5$
   $x \equiv -2 \pmod 7$
   $x \equiv 7 \pmod 2$
   $0 \leq x < 70$

8. $t \equiv 2 \pmod 3$
   $t \equiv 1 \pmod 4$
   $t \equiv -1 \pmod 5$
   $0 \leq t < 60$

9. $z \equiv 3 \pmod 2$
   $z \equiv 1 \pmod 3$
   $z \equiv -1 \pmod 5$
   $z \equiv -4 \pmod 7$
   $0 \leq z < 210$

10. $q \equiv 1 \pmod 3$
    $q \equiv 2 \pmod 4$
    $q \equiv -5 \pmod 5$
    $q \equiv 5 \pmod 7$
    $0 \leq q < 420$

11. $q \equiv 3 \pmod 4$
    $q \equiv 5 \pmod 7$
    $q \equiv -3 \pmod{11}$
    $0 \leq q < 502$

12. $e \equiv 3 \pmod 8$
    $e \equiv 6 \pmod 7$
    $e \equiv 2 \pmod 5$
    $0 \leq e < 316$

In the next six problems, tell whether the given list is a reduced residue system modulo 5.

13. $-1, 0, 1, 2$
14. $-2, 0, 2, 4$
15. $2, 4, 6, 8$
16. $4, 6, -4, 2$
17. $7, 14, 21$
18. $2, 4, 6, 8, 12$

B

19. Solve $x \equiv 2 \pmod 8$
    $x \equiv 1 \pmod 7$
    $x \equiv 3 \pmod 6$
    $0 \leq x < 336$.

20. Solve $3x \equiv 9 \pmod 6$
    $2x \equiv 1 \pmod 5$
    $x \equiv 2 \pmod 7$
    $0 \leq x < 210$.

21. Let p, q, and r be distinct primes. Show there exists n such that $p \mid n$, $q \mid n+1$, and $r \mid n+2$.
22. Give a proof of Theorem 4.3 based on the theorems of Sections 1.1 and 1.2.
23. Show that if $(a,b) = (a,c) = (b,c) = 1$ and if $a \mid m$, $b \mid m$, and $c \mid m$, then $abc \mid m$.
24. Suppose $(a,b) = 1$. Show that as x runs through a complete residue system (mod b) and y runs through a complete residue system (mod a), then $ax + by$ runs through a complete residue system (mod ab).
25. Prove by induction on n that if $b_1, b_2, \dots, b_n$ are integers, relatively prime in pairs, and if $b_i \mid m$ for $i = 1, 2, \dots, n$, then $b_1 b_2 \cdots b_n \mid m$.
26. Prove the previous problem using the fundamental theorem of arithmetic.

The next four problems complete the proof of the Chinese remainder theorem. Assume the notation and assumptions of the theorem in them.

27. Show that the congruence $B_i x_i \equiv 1 \pmod{b_i}$ is solvable for $i = 1, 2, \dots, n$.

28. Show that the number z defined in the theorem satisfies the original system of congruences.
29. Show that if z is a solution to the system and $z' \equiv z \pmod{B}$, then z' is also a solution.
30. Show that if z and z' are solutions to the system of congruences, then $z' \equiv z \pmod{B}$.

True-False. In the next six problems tell which statements are true, and give counterexamples for those that are false. Assume that $x_1, x_2, \ldots, x_t$ is a reduced residue system modulo b, and that $(a, b) = 1$.

31. The list $2x_1, 2x_2, \ldots, 2x_t$ is a reduced residue system (mod b).
32. The list $x_1 + 2, x_2 + 2, \ldots, x_t + 2$ is a reduced residue system (mod b).
33. The list $x_1 + a, x_2 + a, \ldots, x_t + a$ is a reduced residue system (mod b).
34. The list $x_1^2, x_2^2, \ldots, x_t^2$ is a reduced residue system (mod b).
35. The list $-x_1, -x_2, \ldots, -x_t$ is a reduced residue system (mod b).
36. We have $b = \phi(t)$.

37. Give a set of positive integers that is a reduced residue system both modulo 5 and modulo 8.
38. Give a reduced residue system modulo 9 consisting entirely of prime numbers.
39. Professor Crittenden buys a new car every three years; he bought his first in 1981. He gets a sabbatical leave every seven years, starting in 1992. When will he first get both during a leap year?
40. Senator McKinley was first elected in 1980. His reelection is assured unless the campaign coincides with an attack of the seven-year-itch such as hit him in 1983. When must he worry first?
41. A certain baby demands to be fed every five hours. On Tuesday, July 22, its mother watched the start of the NBC *Today* program at 7 A.M. while feeding the baby. When will this happen again?
42. A homeowner drains her hot water heater every 33 days and changes the filter in her heat pump every 56 days. She did the first on January 3, 2000, and the second on January 5, 2000. When after this does she first do both on the same day?
43. On the Fourth of July Sam took a red pill at 8 A.M. and a green pill at noon. The next day he took a white pill at 10 P.M. He takes the red, green, and white pills every 5, 7, and 24 hours, respectively. How often does he take all three together? When in August will this first happen, if at all?
44. Leroy gave each of his four children the same amount of money. One spent all but 13 cents on 5-cent candy bars. (This is a very old problem.) The second spent all but 3 cents on 6-cent iceballs. The third spent all but 2 cents on 11-cent comic books. The fourth bought a \$3 game, but didn't have enough money to buy another. How much money did each child get?