Q1]... [10 points] Sketch the parametric curve

$$x = \cos(t) + \sin(t), \qquad y = \cos(t) - \sin(t), \qquad 0 \le t \le 2\pi$$

by finding a suitable cartesian equation satisfied by x and y. Indicate the point where $t = 0$ and the direction of increasing t on your diagram. Also, find points on the curve which have horizontal tangent lines, and the points which have vertical tangent lines.

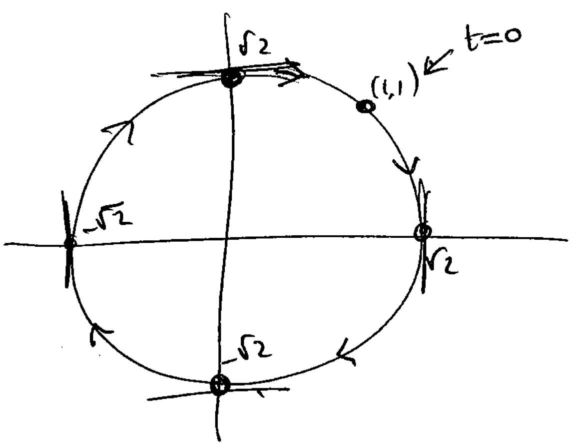
$x^2 + y^2 = \cos^2 t + \sin^2 t + \cancel{2\cos t\sin t} + \cos^2 t + \sin^2 t \cancel{- 2\cos t \sin t} = 1 + 1 = 2$

So points lie on the circle $\boxed{x^2+y^2=2}$. Is the curve the entire circle?

Note: $x = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right) = \sqrt{2}\left(\cos\frac{\pi}{4}\cos t + \sin\frac{\pi}{4}\sin t\right) = \sqrt{2}\cos\left(\frac{\pi}{4} - t\right)$

$y = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos t - \frac{1}{\sqrt{2}}\sin t\right) = \sqrt{2}\left(\sin\frac{\pi}{4}\cos t - \cos\frac{\pi}{4}\sin t\right) = \sqrt{2}\sin\left(\frac{\pi}{4} - t\right)$

as $0 \le t \le 2\pi$
$\Rightarrow$ describe circle in clockwise fashion starting at $(1,1)$



Horizontal Tangents $\frac{dy}{dt} = 0$

$-\sin t - \cos t = 0 \Rightarrow \cos(t) = -\sin(t)$
$\Rightarrow t = 3\pi/4,\ 7\pi/4$

$(0,-\sqrt{2}),\ (0,\sqrt{2})$

Vertical Tangents $\frac{dx}{dt} = 0$

$-\sin t + \cos t = 0 \qquad \cos t = \sin t \qquad t = \pi/4,\ 5\pi/4$

$(\sqrt{2},0) \qquad (-\sqrt{2},0)$

Use the parametric arc length integral to find the length of the curve above. Does your answer make intuitive sense?

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-\sin t + \cos t)^2 + (-\sin t - \cos t)^2}\, dt$$

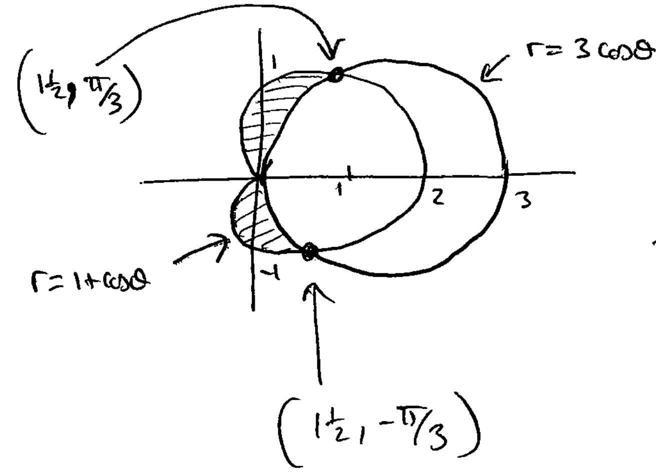
$$= \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}t\Big]_0^{2\pi} = 2\pi(\sqrt{2})$$

$$= 2\pi\,(\text{radius})$$

circle of radius $\sqrt{2}$

yes – Makes sense.

Q2]... [15 points] Find the points of intersection of the cardiod $r = 1 + \cos(\theta)$ and the circle $r = 3\cos(\theta)$, and sketch these two polar curves.

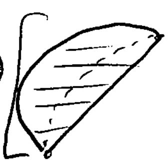


$$3\cos\theta = 1+\cos\theta$$
$$2\cos\theta = 1$$
$$\cos\theta = \tfrac{1}{2} \qquad \theta = \pm \pi/3$$

Intersection points are $\left(1\frac{1}{2}, \pi/3\right)$ and $\left(1\frac{1}{2}, -\pi/3\right)$

Find the area of the region which lies inside the cardiod $r = 1 + \cos(\theta)$ and outside of $r = 3\cos(\theta)$.

Area = Area of shaded region in sketch above. = 2 (Area of shaded region above x-axis)

$= 2\Big[$  $-$  $\Big]$

$$= 2\left[\int_{\pi/3}^{\pi} \tfrac{1}{2}(1+\cos\theta)^2\, d\theta - \int_{\pi/3}^{\pi/2} \tfrac{1}{2}(3\cos\theta)^2\, d\theta\right]$$

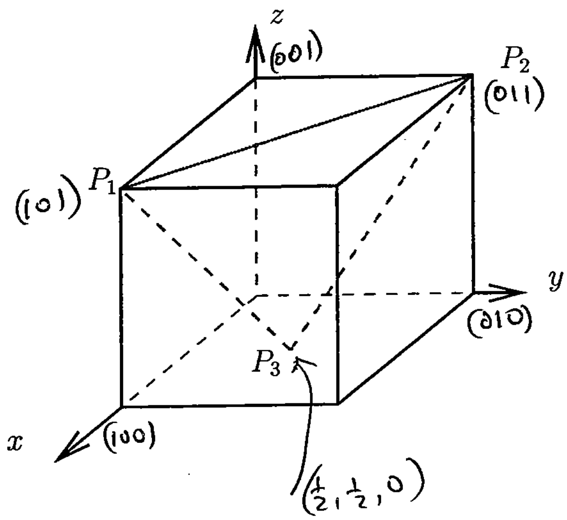
$$= \int_{\pi/3}^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta - \int_{\pi/3}^{\pi/2} 9\cos^2\theta\, d\theta$$

$$= \int_{\pi/3}^{\pi} \left(1 + 2\cos\theta + \frac{\cos(2\theta)+1}{2}\right) d\theta - \int_{\pi/3}^{\pi/2} \frac{9}{2}\left(\cos(2\theta)+1\right) d\theta$$

$$= \left[\theta + 2\sin\theta + \frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right]_{\pi/3}^{\pi} - \left[\frac{9}{4}\sin(2\theta) + \frac{9\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$\leftarrow$ Leave this! $\nearrow$

Q3]... [10 points] Form a triangle by taking one side to be the diagonal P_1P_2 of a face of a unit cube, and by taking the third vertex P_3 to be the center of the opposite face of the cube. See the diagram.



Write down vectors corresponding to the sides P_3P_1 and P_3P_2 of the triangle.

$$\overrightarrow{P_3P_1} = \vec{P_1} - \vec{P_3} = \langle 1,0,1\rangle - \langle \tfrac{1}{2},\tfrac{1}{2},0\rangle = \langle \tfrac{1}{2},-\tfrac{1}{2},1\rangle$$

$$\overrightarrow{P_3P_2} = \vec{P_2} - \vec{P_3} = \langle 0,1,1\rangle - \langle \tfrac{1}{2},\tfrac{1}{2},0\rangle = \langle -\tfrac{1}{2},\tfrac{1}{2},1\rangle$$

This is the dihedral angle in a regular tetrahedron

$$\boxed{\cos^{-1}\left(\tfrac{1}{3}\right)}$$

Find the cosine of the angle at the vertex P_3 in the triangle $P_1P_2P_3$.

$$\cos\theta = \frac{\overrightarrow{P_3P_1}\cdot\overrightarrow{P_3P_2}}{|\overrightarrow{P_3P_1}|\,|\overrightarrow{P_3P_2}|}$$

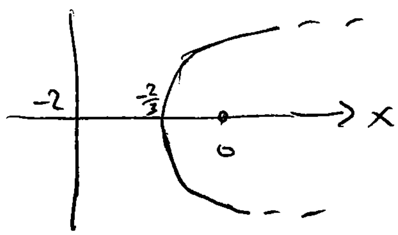
$$= \frac{\langle \tfrac{1}{2},-\tfrac{1}{2},1\rangle \cdot \langle -\tfrac{1}{2},\tfrac{1}{2},1\rangle}{\sqrt{\tfrac{1}{4}+\tfrac{1}{4}+1}\;\sqrt{\tfrac{1}{4}+\tfrac{1}{4}+1}}$$

$$= \frac{-\tfrac{1}{4}-\tfrac{1}{4}+1}{3/2} = \tfrac{1}{2}\cdot\tfrac{2}{3} = \tfrac{1}{3}$$

Q4]... [15 points] For each of the following three polar curves, say which conic section it represents, and write down the eccentricity, one focus, one vertex and one directrix.

$$r = \frac{2}{2-\cos(\theta)}$$

$$r = \frac{1}{1-\frac{1}{2}\cos\theta}$$



(vertices at $\theta = 0, \pi$)

Eccentricity!

$0 < \varepsilon = \frac{1}{2} < 1$

$\Rightarrow$ Ellipse

one focus $= 0$

directrix: $1/\frac{1}{2} = 2$

directrix line is $\boxed{x = -2}$

Vertex is $(-2/3, 0)_{\text{cartesian}}$

$$r = \frac{2}{1+2\cos(\theta)}$$

(vertices at $\theta = 0, \pi$)

Eccentricity $= 2 > 1$

Hyperbola,

one focus $= 0$

directrix $(x = 2/2)$ $\boxed{x = 1}$

Vertex $= (2/3, 0)_{\text{cartesian}}$

$$r = \frac{1}{2-2\sin(\theta)}$$

$$r = \frac{\frac{1}{2}}{1 - 1\sin(\theta)}$$

(Vertices at $\theta = \pi/2, \frac{3\pi}{2}$.)

Eccentricity $= 1$

PARABOLA

one focus $= 0$

directrix $\Rightarrow \boxed{y = -\frac{1}{2}}$

Vertex $= (0, -\frac{1}{4})_{\text{CART.}}$

Q5]...[5 points] Let $\mathbf{u}$ and $\mathbf{v}$ be two non-parallel vectors. Prove that the vector

$$\frac{\mathbf{u}}{|\mathbf{u}|} + \frac{\mathbf{v}}{|\mathbf{v}|}$$

bisects the angle between $\mathbf{u}$ and $\mathbf{v}$. Your argument should work in general (do not just show this works for two specific vectors).

$$\vec{W} \overset{\text{call this } W}{=} \frac{\vec{u}}{|u|} + \frac{\vec{v}}{|\vec{v}|}$$ lives in the same plane as $\vec{u}, \vec{v}$

Therefore, it bisects angle between $\vec{u}$ and $\vec{v}$ if and only if it makes equal angles with $\vec{u}$ & with $\vec{v}$.

This is true if & only if cosine of angle $\frac{\vec{u}}{|u|} + \frac{\vec{v}}{|v|}$ makes with $\vec{u}$ = cosine of angle $\frac{u}{|u|} + \frac{v}{|v|}$ makes with $\vec{v}$.

1st cosine $$= \frac{\left(\frac{\vec{u}}{|u|} + \frac{\vec{v}}{|v|}\right) \cdot \vec{u}}{|(\vec{W})||\vec{u}|}$$

$$= \frac{\frac{|u|^2}{|u|} + \frac{u \cdot v}{|v|}}{|(\vec{W})| \, |\vec{u}|} = \frac{\frac{|u|^2}{|u^2|} + \frac{u \cdot v}{|u||v|}}{|(\vec{W})|} = \frac{1 + \frac{u \cdot v}{|u||v|}}{|(\vec{W})|}$$

2nd cosine $$= \frac{\left(\frac{\vec{u}}{|u|} + \frac{\vec{v}}{|v|}\right) \cdot \vec{v}}{|(\vec{W})| \, |\vec{v}|} = \frac{\frac{u \cdot v}{|u|} + \frac{|v|^2}{|v|}}{|(\vec{W})| \, |\vec{v}|}$$

$$= \frac{\frac{\vec{u} \cdot \vec{v}}{|u||v|} + \frac{|v|^2}{|v|^2}}{|\vec{W}|} = \frac{\frac{u \cdot v}{|u||v|} + 1}{|\vec{W}|}$$

SAME !!

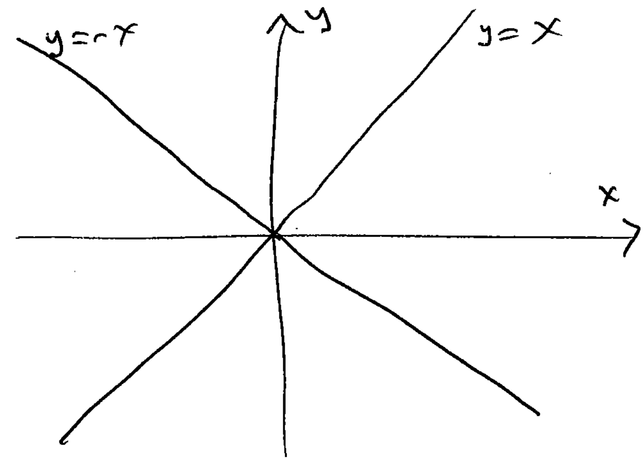
Q6]... [10 points] Show that the equation $y^2 = x^2$ represents two lines in the xy-plane. Sketch these lines.

$y^2 - x^2 = 0$

$(y-x)(y+x) = 0$

$y - x = 0$ $\quad$ $y + x = 0$

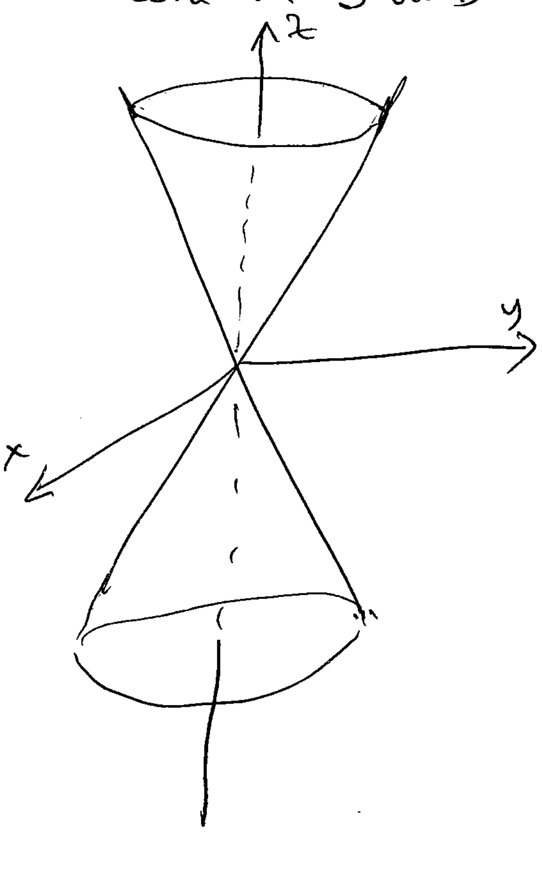
$y = x$ $\quad$ $y = -x$



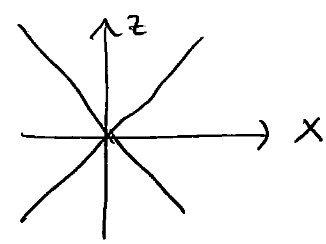
Using the intuition you developed in the first part above, now describe (and sketch a picture) the locus of points in 3–dimensions which satisfy the equation

$$z^2 = x^2 + y^2$$

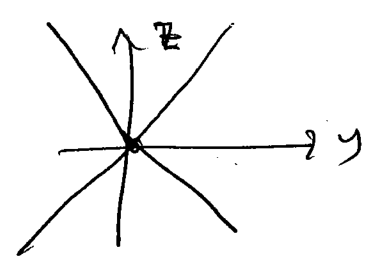
Locus is a cone in 3-dims $\Leftarrow$



① In the vertical xz-plane $(y=0)$ locus gives a pair of lines



② In the vertical yz-plane $(x=0)$ Locus gives a pair of lines



③ On the horizontal plane $z = c$ locus gives a circle

$x^2+y^2 = c^2$

of radius c.

Q7]...[10 points] Find the area of the surface of revolution obtained by revolving the parametric curve

$$x = \cos^3(t), \qquad y = \sin^3(t), \qquad 0 \le t \le \pi/2$$

about the x-axis.

$$\text{Surf. Area} = \int 2\pi(\text{radius})\, ds$$

$$\frac{dx}{dt} = 3\cos^2(t)(-\sin t)$$

$$\frac{dy}{dt} = 3\sin^2(t)(\cos t)$$

$$= 2\pi \int y \, ds$$

$$= 2\pi \int_0^{\pi/2} \sin^3(t)\sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}\, dt$$

$$= 2\pi \int_0^{\pi/2} \sin^3(t)\sqrt{9\cos^2 t\sin^2 t\,\underbrace{(\cancel{\cos^2 t+\sin^2 t})}_{\to 1}}\, dt$$

$$= 2\pi \int_0^{\pi/2} \sin^3(t)\, 3\,|\cos t|\,|\sin t|\, dt$$

$$= 6\pi \int_0^{\pi/2} \sin^3(t)\cos t\sin t\, dt \quad \text{since } 0 \le t \le \pi/2$$

$$= 6\pi \int_0^{\pi/2} \sin^4 t\cos t\, dt$$

$$u = \sin t$$

$$= 6\pi \int_0^1 u^4\, du$$

$$= 6\pi \left.\frac{u^5}{5}\right]_0^1 = \boxed{\frac{6\pi}{5}}$$