MATH 2513-001 FA'09 Mid III Solutions

**Q1]... [20 points]** Define what it means for a function $f : A \to B$ to be surjective.

$f : A \to B$ is <u>SURJECTIVE</u> if $\forall b \in B \ \exists a \in A$ such that $f(a) = b$.

Recall that $(0,1)$ denotes the interval $\{x \in \mathbb{R} \,|\, 0 < x < 1\}$, and that $\mathbb{Z}^+$ denotes the set of positive integers. Give a detailed proof that no function

$$f : \mathbb{Z}^+ \to (0,1)$$

can be surjective. (This is the usual Cantor diagonalization argument that $(0,1)$ is uncountable).

Every $x \in (0,1)$ admits a ! decimal expansion which does not end in an infinite string of 9's. Work with those expansions throughout this proof. ...

Given $f : \mathbb{Z}^+ \to (0,1)$ write out the unique decimal expansions for $f(n)$ for all $n \in \mathbb{Z}^+$:

$$\left.\begin{aligned} f(1) &= 0.\,a_{11}\,a_{12}\,a_{13}\,\cdots \\ f(2) &= 0.\,a_{21}\,a_{22}\,a_{23}\,\cdots \\ f(3) &= 0.\,a_{31}\,a_{32}\,a_{33}\,\cdots \\ &\ \vdots \\ f(n) &= 0.\,a_{n1}\,a_{n2}\,a_{n3}\,\cdots \\ &\ \vdots \end{aligned}\right\}$$

here $a_{ij} \in \{0, 1, \ldots, 9\}$ and for any given $i$ the sequence $a_{i1}, a_{i2}, \ldots$ does not end in an infinite string of 9's.

Define a number $b = 0.b_1 b_2 b_3 \cdots$ so that

$$\begin{cases} \bullet\ b_i \neq a_{ii} & \forall i \\ \bullet\ b_i \neq 0 \ \ \forall i & \\ \bullet\ b_i \neq 9 & \text{for infinitely many values of } i \end{cases}$$

Clearly $0.b_1 b_2 \cdots \in (0,1)$ --- we've ruled out $0.000\cdots$ & $0.999\cdots$

& by construction $0.b_1 b_2 \cdots \notin f(\mathbb{Z}^+)$

$\Rightarrow f$ not surjective. ☒

**Q2]…[20 points]** Prove that there is a bijection between the set $\mathbb{Z}$ and the set $\mathbb{Z}^+$.

Eg. $f : \mathbb{Z} \longrightarrow \mathbb{Z}^+ : n \longmapsto \begin{cases} 2n+1 & \text{if } n \geq 0 \\ -2n & \text{if } n < 0 \end{cases}$

is a bijection.

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$f$ is injective: Given $f(x) = f(y)$.

CASE ① : odd output $\Rightarrow 2x+1 = 2y+1$
$\Rightarrow 2x = 2y \Rightarrow x = y$.

CASE ② : even output $\Rightarrow -2x = -2y$
$\Rightarrow x = y$

In either case, $x = y$.
$\Rightarrow f$ injective.

$f$ is surjective: Given $m \in \mathbb{Z}^+$.

CASE ① : $m$ is even
$\Rightarrow m = 2k$ some $k \in \mathbb{Z}^+$
$= -2(-k) = f(-k)$.

Case ② : $m$ is odd
$\Rightarrow m = 2l+1$ some $l \in \mathbb{Z}$
$= f(l)$.

Prove that there is a bijection between the set $\mathbb{Z}^+ \times \mathbb{Z}^+$ and the set $\mathbb{Z}^+$.

Method ①

$$\begin{array}{llll} (1,1) & (1,2) & (1,3) & \cdots \\ (2,1) & (2,2) & & \\ (3,1) & & & \\ \vdots & & & \end{array}$$

indicates a bijection
$\mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$.

But it takes a lot of care to prove that one exists from this perspective.

Step ① Let $A_n = \{ (p,q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid p + q = n+1 \}$

$A_1 = \{(1,1)\}$, $A_2 = \{(1,2),(2,1)\}$, $A_3 = \{(1,3),(2,2),(3,1)\}$

Note $|A_n| = n$ & $A_n$ are disjoint sets.
& $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}^+ \times \mathbb{Z}^+$.

Step ② Let $B_n = \left\{ \frac{n(n-1)}{2} + 1, \ldots, \underbrace{\frac{n(n-1)}{2} + n}_{} \right\} \subseteq \mathbb{Z}^+$

Note: $\frac{n(n-1)}{2} + n = \frac{(n+1)(n)}{2}$

$B_1 = \{1\}$, $B_2 = \{2, 3\}$, $B_3 = \{4, 5, 6\}$, $B_4 = \{7, 8, 9, 10\}$

Note $|B_n| = n$, the $B_n$ are disjoint and $\bigcup_{n=1}^{\infty} B_n = \mathbb{Z}^+$.

Since $|A_n| = n = |B_n| \Rightarrow \exists$ bijections

$$A_n \xrightarrow{f_n} \overline{n} \qquad B_n \xrightarrow{g_n} \overline{n}$$

$\Rightarrow g_n^{-1} \circ f_n : A_n \to B_n$ is a bijection.

Now define $F : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ by

$$F\big|_{A_n} = g_n^{-1} \circ f_n$$

Check $F$ is a bijection.

$\boxed{F \text{ surjective}}$ Given $m \in \mathbb{Z}^+$

$\Rightarrow m \in B_n$ for some $n$

$\Rightarrow m = g_n^{-1} \circ f_n(a)$ for some $a \in A_n \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$

--- since $g_n^{-1} \circ f_n$ was bijection

$\Rightarrow m = F(a)$ for some $a \in \mathbb{Z}^+ \times \mathbb{Z}^+$

$\boxed{F \text{ injective}}$ $F(a) = F(b)$

$\Rightarrow F(a) = F(b)$ lie in some $B_n$.

$\Rightarrow a, b$ must lie in $A_n$

& $F(a) = F(b)$

$\Rightarrow g_n^{-1} \circ f_n(a) = g_n^{-1} \circ f_n(b)$

$\Rightarrow a = b$ since $g_n^{-1} \circ f_n$ bij.

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Method ②

Now apply Schröder-Bernstein!

$f : \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+ : n \longmapsto (n, 1)$ is clearly an injection.

$g : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+ : (m, n) \longmapsto 2^m 3^n$ is also an injection.

$$g(m,n) = g(a,b) \Rightarrow 2^m 3^n = 2^a 3^b$$

If $m \neq a$ then $m < a$ (without loss of generality)

$\div$ both sides by $2^m$ to get

$$\underset{\text{odd}}{3^n} = \underset{\text{even}}{2^{a-m} 3^b} \Rightarrow \text{contradiction!}$$

$\Rightarrow m = a$

$m = a \Rightarrow \div$ both sides by $2^m$ to get $3^n = 3^b$

$\Rightarrow n = b$ (take $\log_3(\cdot)$).

$\Rightarrow (m, n) = (a, b)$

$\Rightarrow g$ injective.

**Q3]... [24 points]** Define what it means for the set $A$ to be countable.

$A$ is countable if $A$ is finite <u>OR</u> $\exists$ bijection $\mathbb{Z}^+ \to A$.

Define what it means for the set $A$ to be uncountable.

$A$ is uncountable if $A$ is <u>not</u> countable. This means, $A$ is not finite <u>and</u> $\nexists$ bijection $\mathbb{Z}^+ \to A$.

Say whether each of the following sets is countable or not.

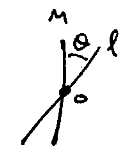
1. The set $\mathbb{Z}$ of all integers. C (see Q2 !)
2. The set $\mathbb{R}$ of all real numbers. U (see Q1 !)
3. The set $\mathbb{Q}$ of all rational numbers. C (see Q2 ! )
4. The set of all irrational numbers. U (since $\{\text{irrational}\} \cup \mathbb{Q} = \mathbb{R}$)
5. The set of all functions from $\{1\}$ to $\mathbb{R}$. $\xleftrightarrow[\text{bij.}]{} \mathbb{R} \Rightarrow$ U
6. The set of all functions from $\mathbb{R}$ to $\{1\}$. $\xleftrightarrow[\text{bij.}]{} \{1\} \Rightarrow$ finite $\Rightarrow$ countable, C.
7. The set of all functions from $\{1, 2, 3\}$ to $\mathbb{Z}$. $\xleftrightarrow[\text{bij.}]{} \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \Rightarrow$ Countable, C.
8. The set of all functions from $\mathbb{Z}$ to $\{1, 2, 3\}$. U do diagonal argument.
9. The set of all lines in the plane $\mathbb{R}^2$. $\xleftrightarrow[\text{bij}]{} \mathbb{R} \times \mathbb{R} \Rightarrow$ U
10. The power set $\mathbb{P}(\mathbb{Z})$ of $\mathbb{Z}$. U

from class notes.

Line $\Rightarrow$ $y = mx + b$ for some $m \in \mathbb{R}$, $b \in \mathbb{R}$

$$\{y = mx + b \mid m, b \in \mathbb{R}\} \xleftrightarrow[\text{bij.}]{} \{(m, b) \mid m, b \in \mathbb{R}\} = \mathbb{R}^2$$

$m\,\ell$ = Rotation about $O$ through $2\theta$.




**Q4]... [36 points]** True or False.

1. The composition of reflections in two intersecting lines is a rotation. T

2. The set of symmetries of a regular pentagon (5 sides) has 10 elements. T
   5 reflections
   5 rotations

3. The set of symmetries of a regular polygon with 1,000 sides is countable. T
   has 2,000 elements !

4. The set of symmetries of a circle is countable. F (has rotations through $\theta$ $\forall \theta \in [0, 2\pi)$) ✓

5. Perm($\{1, 2, \ldots, n\}$) has $n^n$ elements. F has $n!$ elements

6. Perm($\mathbb{Z}^+$) is countable. F (see below (*))

7. $(123)(245)(132) = (345)$. T (just do it!)

8. If $m$ is reflection in some line, and $R$ is a 90° counterclockwise rotation about a point $O$, then $mRm$ is a 90° counterclockwise rotation about the point $m(O)$. F. It is a <u>clockwise</u> rotation about $m(O)$.

9. $(12)(23)(34)(45)(56)(67)(78) = (12345678)$ T (just do it!)

10. $(12)(23)(34)(45)(34)(23)(12) = (15)$. T (just do it!)

11. The composition of reflections in the three sides of a triangle (taken in any order) is a rotation. F (odd # of reflections in lines can never yield a rotation!)

12. The composition of reflections in the four sides of a rectangle (taken in any order) is a translation. T

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(*) Let $\sigma_1 = (12)$, $\sigma_2 = (34)$ ---, $\sigma_n = (2n-1, 2n)$, ----- $\in$ Perm$(\mathbb{Z}^+)$

Then Perm$(\mathbb{Z}^+)$ contains all $\infty$ strings of the form

$$\sigma_1^{a_1}\sigma_2^{a_2}\sigma_3^{a_3} \text{----} \quad \text{where } a_i \in \{1, 2\}$$

& we can do a diagonal argument to show this set is uncountable. //