Calc I — Fall 2010 — Mid III (sol^{ns})

Q1]...[15 points] Suppose that the derivative of a function f is

$$f'(x) = (x+1)(x-3)^2(x-6)^5$$

Write down the critical points of f.

where $f'(x)=0$ | $x+1=0$, $x=-1$ | $(x-3)^2=0$, $(x-3)=0$, $x=3$ | $(x-6)^5=0$, $(x-6)=0$, $x=6$

Find the intervals on which f is increasing, and the intervals on which f is decreasing. For each critical point of f, say whether it is a local maximum, a local minimum, or neither.

Interval	$(-\infty,-1)$	$(-1,3)$	$(3,6)$	$(6,\infty)$
Sign of $f'(x)$	$+$	$-$	$-$	$+$
Behavior of $f(x)$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$

Local Max at -1

Neither local max nor local min at 3

Local Min at 6

Q2]... **[15 points]** State the Mean Value Theorem (MVT).

Suppose $f(x)$ is differentiable on (a,b) & cts. on $[a,b]$.
Then there exists a point c in (a,b) so that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Use the MVT to show that if $f'(x) > 0$ for all x on an interval I, then $f(x)$ is increasing on I.

Given two input points $x_1 < x_2$ in the interval I,
we want to conclude that $f(x_1) < f(x_2)$.

MVT $\Rightarrow$ $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$ for some c between x_1 & x_2

> 0

but then c is in I
& so $f'(c) > 0$

fraction > 0

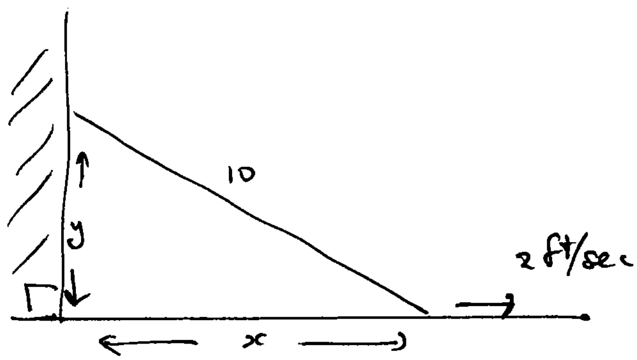
& denominator > 0

$\Rightarrow$ Numerator > 0

$\Rightarrow$ $f(x_2) > f(x_1)$

done!

Q3]... [20 points] A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 ft per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 8 ft from the wall? Show the steps of your work.



$$\boxed{x^2 + y^2 = 10^2} \quad \text{——} (*)$$

Told $\frac{dx}{dt} = 2$

Asked for $\frac{dy}{dt}$ (when $x = 8$)

$x = 8 \Rightarrow 64 + y^2 = 100$
$\Rightarrow y^2 = 36$
$\Rightarrow y = 6$

Also $\frac{d}{dt}(*) \Rightarrow$

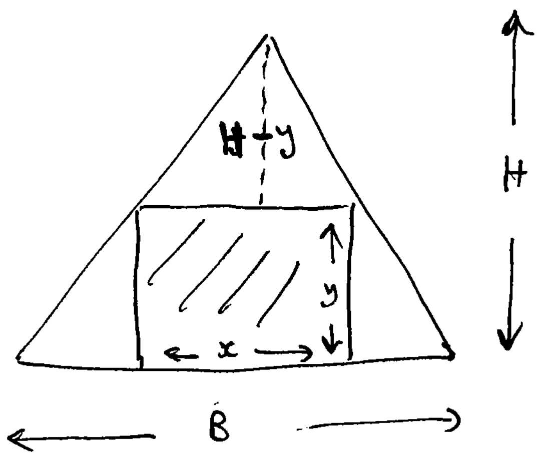
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = \frac{d}{dt}(100) = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{8}{6}2$$

$$= -\frac{8}{3} \text{ ft/sec}$$

Q4]... **[20 points]** Find the dimensions of the rectangle of largest area that can be inscribed inside of an isosceles triangle with base length B and height H. Your answer will involve B and H. Show the steps of your work.



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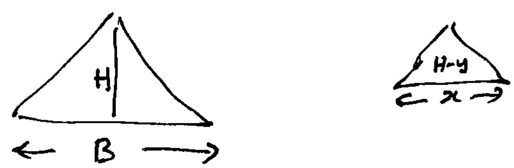


$$\frac{x}{B} = \frac{H-y}{H}$$

$$x = \left(\frac{H-y}{H}\right)B$$

$$\text{Area} = xy$$

$$= \left(\frac{H-y}{H}\right)B\,y$$

$$A(y) = \frac{B}{H}\left(Hy - y^2\right)$$ .... Remember H, B are fixed constants!

$$A'(y) = \frac{B}{H}\left(H.1 - 2y\right)$$

At max $\quad A'(y) = 0 \Rightarrow H - 2y = 0 \Rightarrow y = H/2 \leftarrow$

$\Rightarrow x = B/2 \leftarrow$) dimensions

$$\& \text{ Area} = \frac{HB}{4}$$

$$= \frac{1}{2}(\text{Area of triangle})$$

Q5]...[15 points] Find the linearization of $f(x) = \sqrt[3]{1+3x}$ at the point $a = 0$.

$$L(x) = f(0) + f'(0)(x-0)$$

$$= 1 + 1 \cdot x$$

$$\boxed{L(x) = 1 + x}$$

$$f(0) = \sqrt[3]{1} = 1$$

$$f'(x) = \frac{1}{3}(1+3x)^{-2/3} \cdot 3$$

$$f'(0) = \frac{1}{3}(1)^{-2/3} \cdot 3$$

$$= 1$$

USe the linearization above to give an approximate value for $\sqrt[3]{1.03}$.

$$\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)}$$

$$= f(0.01)$$

$$\approx L(0.01)$$

$$= 1 + 0.01$$

$$= 1.01$$

Q6]... **[15 points]** Find the critical points for the function $f(x) = 4x - \tan(x)$ in the interval $[0, \pi]$.

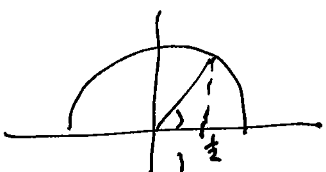
$$f'(x) = 4 - \sec^2(x)$$

$$f'(x) = 0 \Rightarrow 4 - \frac{1}{\cos^2(x)} = 0$$

$$\Rightarrow \cos^2(x) = \frac{1}{4}$$

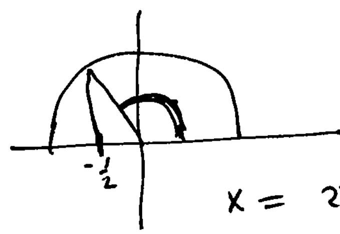
$$\Rightarrow \cos(x) = \pm \frac{1}{2}$$

$\cos x = \frac{1}{2}$ & $0 \leq x \leq \pi$



$x = \pi/3$

$\cos x = -\frac{1}{2}$ & $0 \leq x \leq \pi$



$x = 2\pi/3$

Ans: $\pi/3$ & $2\pi/3$