Precise Definition of a Limit

We have been developing an intuition about limits, and have a working "definition" in hand. We had looked at the following version. This is taken from the "Roughly speaking" paragraph on page 50 of your book.

Version 1. *Suppose $f(x)$ is a function which is defined on an interval about a, but possibly not at a. Define $\lim_{x\to a} f(x) = L$ if the values of $f(x)$ get closer and closer to L as the x values get closer and closer to a, but $x \neq a$.*

We could define one sided limits similarly.

Objection 1. However, one could point out that the values of the function $f(x) = \frac{1}{x}$ get closer and closer to 0 as x increases from 1 to $1,000,000$. Yet you would not write

$$\lim_{x\to 1{,}000{,}000^+} \frac{1}{x} = 0$$

in this instance.

The book gets around this objection by using the phrase "arbitrarily close" in place of the first "closer and closer" and the phrase "sufficiently close" in place of the second "closer and closer."

Version 2. *Suppose $f(x)$ is a function which is defined on an interval about a, but possibly not at a. Define $\lim_{x\to a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x values sufficiently close to a, but $x \neq a$.*

Objection 2. This is better than version 1. However, the word "close" is still vague. What do we mean by "arbitrarily close" and by "sufficiently close"?

Ideas of closeness come up in real life whenever we try to measure something. We work with "approximations" and use "error tolerances" which are very much context specific. Consider the following list of questions.

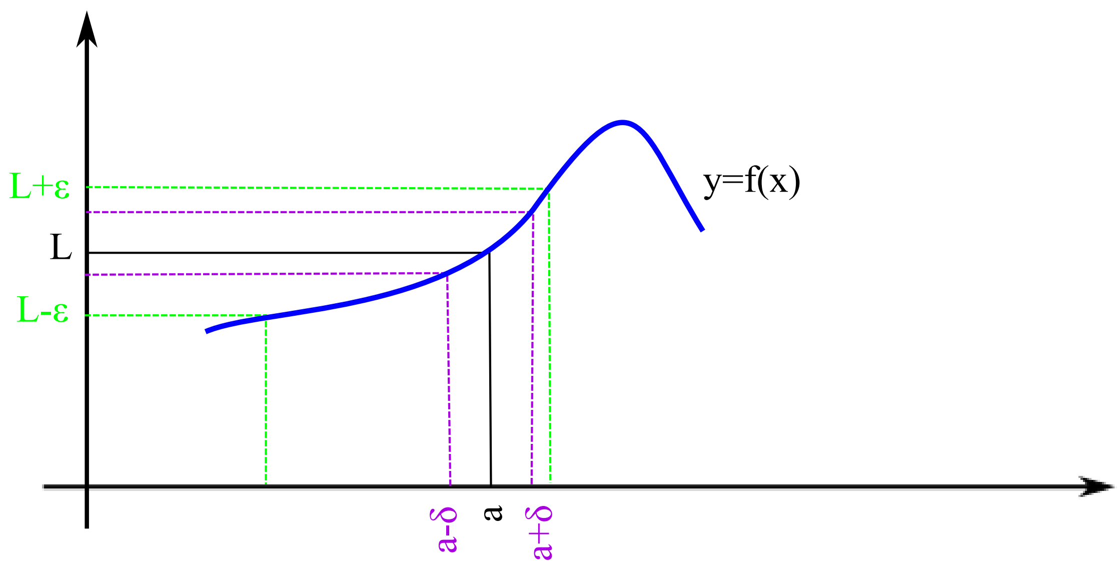
1. How far is it from here to the surface of the moon?
2. How far is it from here to the Devon tower in OKC?
3. How far is it from here to the main library on campus?
4. How far is it from the front of the classroom to the back of the classroom?
5. How far is it from one corner of your textbook to the diametrically opposite corner?

You will not know precise answers to any of these questions. But you can give good estimates and will back them up with error tolerances (using phrases like "give or take" etc.). In each case your error tolerances might be in 1,000's of miles; in miles; in yards; in feet; in inches.

An engineer who is designing a component for some high speed machine may work with tolerances of $\frac{1}{1{,}000}$ of an inch. A nuclear physicist may come up to you and the engineer deep in discussion about the machine part and say (with a wry smile), *"I couldn't help but overhear you two discussing fine tolerances . . . "*

The mathematician overhears this conversation and has an epiphany. Everyone has their own notion of tolerance. If the mathematician is to claim that a limit equals L, then she must ensure that everyone's notion of output tolerance is satisfied. This can be phrased as a guarantee: If you hand the mathematician a number $\epsilon > 0$ which indicates your accepted notion of tolerance for the outputs, she will return a number $\delta > 0$ which when used as an input tolerance around a will give outputs within ϵ of L.

Version 3. *Suppose $f(x)$ is a function which is defined on an interval about a, but possibly not at a. Define $\lim_{x\to a} f(x) = L$ if for every notion of output tolerance $\epsilon > 0$, there is a corresponding notion of input tolerance $\delta > 0$ so that if x is within δ of a, but not equal to a, then $f(x)$ is within ϵ of L.*



Graphical interpretation of the definition of a limit. In our universe, graphs are blue, input tolerances are purple and output tolerances are green!

Now the mathematician realizes that the phrase "notion of output tolerance" is not necessary, since the phrase "then $f(x)$ is within ϵ of L" later on in the definition clearly indicates that ϵ is an output tolerance. Similarly, the phrase "notion of input tolerance" is redundant, since the phrase "if x is within δ of a" later in the definition means that δ is being used as an input tolerance.

The mathematician knows that the distance between two numbers p and q is just their difference $(p - q)$ provided it is given a positive sign, that is, provided she takes the absolute value $|p - q|$. Using this notion of distance, the phrase "$f(x)$ is within ϵ of L" translates as $|f(x) - L| < \epsilon$, and the phrase "x is within δ of a, but not equal to a" translates as $0 < |x - a| < \delta$. Note that the $0 < |x - a|$ portion means that x is a positive distance from a and so $x \neq a$.

Final touches. You may see the symbol $\forall$ used for the phrase "for all," and the symbol $\exists$ for the phrase "there exists."

Combining the reductions/translations of the last 3 paragraphs, and invoking the mathematician's mantra — *More symbols good, More words bad!* — one obtains the following definition.

Version 4. *Suppose $f(x)$ is a function which is defined on an interval about a, but possibly not at a. Define $\lim_{x\to a} f(x) = L$ if $\forall\epsilon > 0$, $\exists\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.*