Math 1914. Extra Hwk II   Derivatives Without Difference Quotients!

Q1. Derivative as a Linear Approximation.

- We saw in class that if $f'(a)$ exists, then the following expression holds

$$f(a+h) \;=\; f(a) + f'(a)h + \epsilon h$$

  where $\epsilon \to 0$ as $h \to 0$.

- We also saw that if the function $f(x)$ satisfies the following

$$f(a+h) \;=\; f(a) + Lh + \epsilon h \qquad\qquad (*)$$

  for some number L and where $\epsilon \to 0$ as $h \to 0$, then the function is differentiable at the input a and $f'(a) = L$.

- So the expression $(*)$ together with the condition that $\epsilon \to 0$ as $h \to 0$ is another way to say that the function $f(x)$ is differentiable at the input a. Notice that there are no "difference quotients" in sight.

- So if $f(x)$ is differentiable at the input a the straight line $y = f(a) + f'(a)h$ approximates the values of $f(a+h)$ with an error term of ϵh which tends to 0 as $h \to 0$ faster than $h \to 0$. We see this because ϵh is the product of two quantities that are both going to 0 as $h \to 0$, and one of them is h.

  Note that the expression $y = f(a)+f'(a)h$ really is a line. If we want to write things explicitly in terms of x we remember that $x = a+h$. Thus $h = (x-a)$ and the linear approximation is just the tangent line

$$y \;=\; f(a) + f'(a)(x-a)\,.$$

This first question asks you to check how well $f(a)+f'(a)h$ approximates $f(a+h)$ in certain cases.

(a) Let $f(x) = \sqrt{x}$, and $a = 4$. Write out the linear approximation for $\sqrt{4+h}$ in this case.

(b) Compare your output from the linear approximation for $\sqrt{4.001}$ with the actual value of $\sqrt{4.001}$. Do the same for $\sqrt{4.0001}$.

(c) Consider the function $f(x) = \frac{1}{1+x}$ at the input 1. Use the linear approximation definition of the derivative to show that

$$\frac{1}{1+u} \;=\; 1 - u + \epsilon u \qquad\qquad (**)$$

where $\epsilon \to 0$ as $u \to 0$.

(d) Use the expression you obtained above to approximate $\frac{1}{1.0004}$. Compare your answer with the actual value of $\frac{1}{1.0004}$.

(e) Use the expression obtained above to find an approximation for $\frac{1}{3.006}$. Compare your answer with the actual value of $\frac{1}{3.006}$.

Q2. An Intuitive Proof of the Product Rule. The linear approximation definition of the derivative gives a *simple* and *intuitive* proof of the product rule. It is *simple* because there is nothing to do except multiply out two expressions; there is no algebra "magic" involving adding and subtracting the same terms. It is *intuitive* because you see that the $f'g + fg'$ term appears as the cross product terms when multiplying these expressions. It should be "hard to forget" the formula $f'g + fg'$ after this!

Recall, we are told that $f(x)$ and $g(x)$ are each differentiable at the input a, and we have to prove that the product function $f(x)g(x)$ is also differentiable at a and find the expression for its derivative at a.

Step I. Since $f'(a)$ exists, we know that

$$f(a+h) \;=\; f(a) + f'(a)h + \epsilon_1 h \qquad (i)$$

where $\epsilon_1 \to 0$ as $h \to 0$.

Also, since $g'(a)$ exists, we know that

$$g(a+h) \;=\; g(a) + g'(a)h + \epsilon_2 h \qquad (ii)$$

where $\epsilon_2 \to 0$ as $h \to 0$.

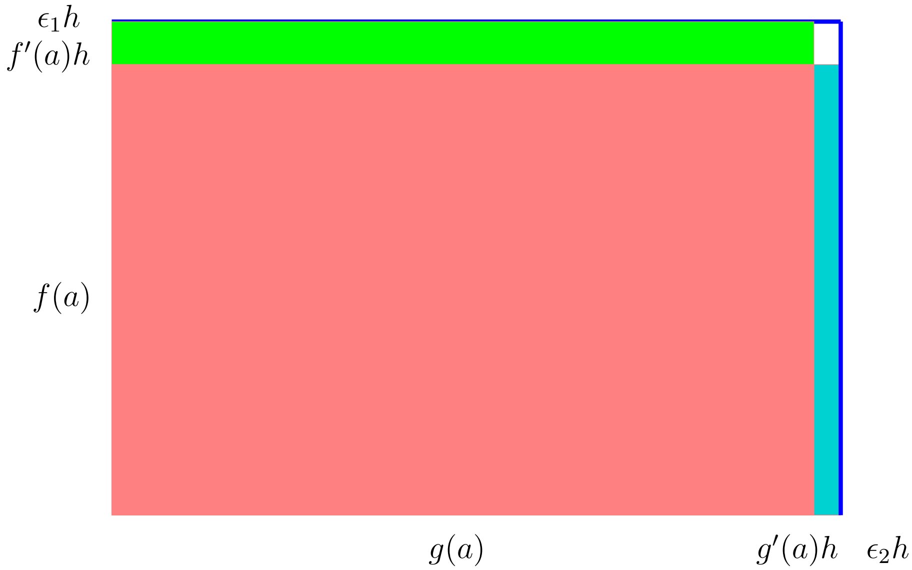
Step II. Now combine the expressions (i) and (ii) to get

$$f(a+h)g(a+h) \;=\; (f(a) + f'(a)h + \epsilon_1 h)(g(a) + g'(a)h + \epsilon_2 h)$$

Multiply these two expressions out and gather terms which have no h, just one h and no ϵ's, and the remaining terms to get

$$f(a+h)g(a+h) \;=\; f(a)g(a) + [f'(a)g(a) + f(a)g'(a)]h + \epsilon_3 h$$

where your ϵ_3 will be a sum of products of other terms. Write out ϵ_3 explicitly and verify that it does indeed tend to 0 as $h \to 0$. What can you conclude from this?



Remark 1. The picture proof of the product rule in terms of areas of rectangles now corresponds to a precise proof using the linear approximation definition of derivative. Look at the picture. The width of the original pink rectangle was $g(a)$ and is now perturbed to $g(a+h)$. We write this new width out as the sum of $g(a)$ (pink rectangle width) plus $g'(a)h$ (cyan rectangle width) plus $\epsilon_2 h$ (vertical blue strip width).

Similarly, the perturbed height $f(a+h)$ is the sum of $f(a)$ (pink rectangle height) plus $f'(a)h$ (green rectangle height) plus $\epsilon_1 h$ (horizontal blue strip height). The picture conveys the fact that the thickness of the blue strips goes to 0 faster than the thickness of the green or cyan rectangles as $h \to 0$.

So this new bigger rectangle is divided into 9 subrectangles; corresponding to the 9 algebra terms that you obtained in step II above. Here they are:

1. the horizontal blue strip (thin rectangle) lying precisely above the green rectangle;
2. the shorter horizontal blue strip (thin rectangle) lying precisely above the white square;
3. the minuscule blue rectangle formed where the horizontal and vertical blue strips intersect;
4. the vertical blue strip precisely to the right of the white rectangle;
5. the vertical blue strip precisely to the right of the cyan rectangle;
6. the white rectangle;
7. the green rectangle;
8. the cyan rectangle; and
9. the pink rectangle.

The first 5 rectangles have area that tends to 0 faster than h (including the white rectangle whose area is proportional to h^2). The green and cyan rectangles have areas which sum to the value $f'(a)g(a) + f(a)g'(a)$; this represents the first order (or linear) change in the area of the original rectangle. The original pink rectangle has area $f(a)g(a)$.

Q3. A Somewhat Intuitive Proof of the Quotient Rule. The linear approximation definition of the derivative can also be used to give a proof of the quotient rule. You will start exactly as in the previous question. The problem you will encounter is that the $g(a+h)$ term is below the line. However, we have seen in **Q1(c)** that the linear approximation version of the derivative allows us to deal with reciprocals of sums...

Recall that we are told that $f(x)$ and $g(x)$ are differentiable at the input a and that $g(a) \neq 0$, and we have to conclude that $\frac{f}{g}$ is differentiable at the input a and we have to establish the quotient rule formula for its derivative.

Step I. Since $f'(a)$ exists, we know that

$$f(a+h) \;=\; f(a) + f'(a)h + \epsilon_1 h \qquad (i)$$

where $\epsilon_1 \to 0$ as $h \to 0$.

Also, since $g'(a)$ exists, we know that

$$g(a+h) \;=\; g(a) + g'(a)h + \epsilon_2 h \qquad (ii)$$

where $\epsilon_2 \to 0$ as $h \to 0$.

Step II. Now combine the expressions (i) and (ii) to get

$$\frac{f(a+h)}{g(a+h)} \;=\; \frac{f(a) + f'(a)h + \epsilon_1 h}{g(a) + g'(a)h + \epsilon_2 h}$$

Factor out a $g(a)$ from the denominator to get

$$\frac{[f(a) + f'(a)h + \epsilon_1 h]}{g(a)[1 + \frac{g'(a)}{g(a)}h + \frac{\epsilon_2}{g(a)}h]} \;=\; \frac{1}{g(a)}[f(a) + f'(a)h + \epsilon_1 h]\frac{1}{[1 + \frac{g'(a)}{g(a)}h + \frac{\epsilon_2}{g(a)}h]}$$

Step III. Now denote $\frac{g'(a)}{g(a)}h + \frac{\epsilon_2}{g(a)}h$ by u and use the result $(**)$ from **Q1(c)** to replace the quotient term $\frac{1}{[1+\frac{g'(a)}{g(a)}h+\frac{\epsilon_2}{g(a)}h]}$ by a sum of terms.

Step IV. Multiply out the two expressions and gather together terms which do not involve h, which involve only one h and no ϵ's, and the remaining terms. You should get something that looks like the following

$$\frac{f(a+h)}{g(a+h)} \;=\; \frac{f(a)}{g(a)} + Mh + \epsilon_3 h$$

Verify that the term M is indeed the "formula" for the quotient rule. Write out the expression for ϵ_3 explicitly, and verify that indeed $\epsilon_3 \to 0$ as $h \to 0$. What can you conclude from this?

Remark 2. The quotient rule looks mysterious. There are terms like $f'g$ and $g'f$ as in the product rule, and this proof shows how they arise as cross product terms in products of sums of terms, just as in the case of the product rule. The minus sign may also seem mysterious, but you should remember that it comes from expression $(**)$ for $\frac{1}{1+u}$ in **Q1(c)**. The examples in **Q1(d-e)** should help reenforce the intuition that for small values of u the reciprocal of $(1+u)$ is very nearly equal to $(1-u)$.