MATH 2513–002 Comments on Class Assignment 01

Our first class assignment involved translating propositions about natural numbers into conditional ("if…then …") statements, and then giving careful proofs of these statements. The proofs had the following outline.

1. Start by writing down the hypothesis,
2. Unravel any definitions of terms (such as "even" or 'odd"),
3. Work towards the conclusion (for example which concerns a product of integers, a square or a cube of an integer). This step may involve some algebra manipulations,
4. Finally obtain the conclusion (perhaps using some of the definitions again).

Let us will work through two sentences slowly as examples, using the two column format (first column contains the claims or statements, and the second contains the reasons).

Proposition 1. *The product of two odd integers is an odd integer.*

First of all, we rewrote this as a conditional statement, using explicit variable names for the two integers.

Proposition 1. *If m and n are odd integers, then the product mn is an odd integer.*

Proof.

Claim	**Reason**
Let m and n be odd integers.	…Hypothesis.
Thus $m = 2p + 1$ for some integer p,	…Def. of odd
and $n = 2q + 1$ for some integer q.	…Def. of odd
Therefore $mn = (2p + 1)(2q + 1)$	
$= (2p)(2q) + (2p)(1) + (1)(2q) + (1)^2$	…Algebra manipulations.
$= 4pq + 2p + 2q + 1$	
$= 2(2pq + p + q) + 1.$	
Now $2pq + p + q$ is an integer, which we can denote by k,	…$\mathbb{Z}$ is closed under $\cdot$ and $+$
and so $mn = 2k + 1$ is an odd integer. $\square$	…Def. of odd

Remark. If we were being super-careful, we could break the "algebra manipulations" reason into several concrete steps which involve the basic properties of multiplication and addition which are outlined in Table 1.2 on page 18 of the textbook.

mn	=	(2p+1)(2q+1)	
	=	(2p+1)(2q) + (2p+1)(1)	…Distributive Properties
	=	(2p)(2q) + (1)(2q) + (2p)(1) + (1)(1)	…Distributive Properties
	=	(2p)(2q) + 2q + 2p + 1	…Identity Properties
	=	2(p(2q)) + 2q + 2p + 1	…Associative Properties
	=	2(p(2q)) + 2(q + p) + 1	…Distributive Properties (factoring)
	=	2[p(2q) + q + p] + 1	…Distributive Properties (factoring)
	=	2[(p2)q + q + p] + 1	…Associative Properties
	=	2[(2p)q + p+ q] + 1	…Commutative Properties
	=	2[2(pq) + p + q] + 1	…Associative Properties

This takes a lot of time and energy, and is usually omitted from proofs. However, it is worthwhile doing a computation like this once or twice in your lifetime so you can convince yourself that all the algebra manipulation techniques of multiplying out parentheses (FOIL-ing), factoring etc that you are used to from high school really just follow from the simple laws (identity, inverse, commutativity, associativity and distributivity) which are described in Table 1.2 of your textbook.

Proposition 2. *The product of an integer and an even integer is an even integer.*

Rewriting in conditional form with variable names for the integers gives ...

Proposition 2. *If m is an integer and n is an even integer, then mn is an even integer.*

Proof.

Claim		**Reason**
Let m be an integer and n be an even integer.		... Hypothesis.
Thus $n = 2p$ for some integer p.		... Def. of even
Therefore $mn = m(2p) = 2(mp)$.		... Algebra manipulations.
Now mp is an integer, which we can denote by k,		... $\mathbb{Z}$ is closed under $\cdot$
and so $mn = 2k$ is an even integer.	□	... Def. of even

In this case, the detailed version of the algebra manipulations would be...

$$\begin{aligned} \text{mn} &= \text{m(2p)} && \text{... given that } n = 2p \\ &= \text{(m2)p} && \text{... Associative Properties} \\ &= \text{(2m)p} && \text{... Commutative Properties} \\ &= \text{2(mp)} && \text{... Associative Properties} \end{aligned}$$

Remarks. If you look at your class project, you might notice that there are similarities between your work and the algebra manipulations involved in the proofs of the two propositions above. Indeed, we can now use these two propositions to give "shorter proofs" of some of the other statements in the class projects.

Example 1. *The square of an even integer is even.*

Proof. Let n be even. Then $n^2 = n.n$ is the product of the even integer n with itself. This is even by Proposition 2 with $m = n$. □

Example 2. *The square of an odd integer is odd.*

Proof. Let n be odd. Then $n^2 = n.n$ is the product of the odd integer n with itself. This is odd by Proposition 1 with $m = n$. □

Example 3. *The cube of an odd integer is odd.*

Proof. Let n be odd. Then $n^3 = n^2.n$ is the product of the integer n^2 with the odd integer n. By example 2, we know that since n is odd, then n^2 is also odd. Thus $n^3 = n^2.n$ is odd by Proposition 1 with $m = n^2$. □